



An AM-GM Variation: 10672

V. Anil Kumar; John H. Lindsey II

The American Mathematical Monthly, Vol. 106, No. 10. (Dec., 1999), p. 969.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9890%28199912%29106%3A10%3C969%3AAAV1%3E2.0.CO%3B2-D>

The American Mathematical Monthly is currently published by Mathematical Association of America.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/maa.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

An AM-GM Variation

10672 [1998, 559]. *Proposed by V. Anil Kumar, Kerala Agricultural University, Tavanur, Kerala, India.* Let p_1, p_2, \dots, p_m be positive real numbers summing to 1, and assume that $a_{i,j} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Prove that

$$\sqrt[n]{\prod_{j=1}^n \left(\sum_{l=1}^m p_l a_{l,j} \prod_{i \neq l} \left(\sum_{k=1}^n a_{i,k} \right) \right)} \leq \frac{1}{n} \prod_{i=1}^m \left(\sum_{k=1}^n a_{i,k} \right).$$

Solution by John H. Lindsey II, Fort Meyers, FL. With $x_j = \sum_{l=1}^m p_l a_{l,j} \prod_{i \neq l} \left(\sum_{k=1}^n a_{i,k} \right)$, the left-hand side is the geometric mean of x_1, \dots, x_n and hence is less than or equal to the arithmetic mean of x_1, \dots, x_n , which is

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \left(\sum_{l=1}^m p_l a_{l,j} \prod_{i \neq l} \left(\sum_{k=1}^n a_{i,k} \right) \right) &= \frac{1}{n} \sum_{l=1}^m p_l \left(\sum_{j=1}^n a_{l,j} \prod_{i \neq l} \left(\sum_{k=1}^n a_{i,k} \right) \right) \\ &= \frac{1}{n} \sum_{l=1}^m p_l \prod_{i=1}^m \left(\sum_{k=1}^n a_{i,k} \right) = \frac{1}{n} \prod_{i=1}^m \left(\sum_{k=1}^n a_{i,k} \right). \end{aligned}$$

Solved also by S. Amighibech (France), R. J. Chapman (U. K.), Q. H. Darwish (Oman), W. Janous (Austria), B. Kalantari, S. S. Kim (Korea), M. S. Klamkin (Canada), R. Martin (U. K.), A. Nijenhuis, C. R. Pranesachar (India), H.-J. Seiffert (Germany), S. M. Soltuz (Romania), S.-E. Takahasi (Japan), T. V. Trif (Romania), GCHQ Problems Group (U. K.), and the proposer.

Functions with a Polynomial Addition Formula

10675 [1998, 560]. *Proposed by Harry Tamvakis, University of Pennsylvania, Philadelphia, PA.* Find every continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that some polynomial $P(x, y) \in \mathbb{R}[x, y]$ satisfies $f(x + y) = P(f(x), f(y))$ for every $x, y \in \mathbb{R}$.

Solution by GCHQ Problems Group, Cheltenham, U. K. The function f can take one of two forms:

(i) $f(x) = ax - c$ using $P(u, v) = u + v + c$, including the special case of constant f when $a = 0$; and

(ii) $f(x) = (d^x - a)/b$ using $P(u, v) = a(u + v) + buv + (a^2 - a)/b$.

When $y = 0$, we get $f(x) = P(f(x), f(0)) = Q(f(x))$ for some polynomial Q . If the degree of Q is more than 1, then the value of f is restricted to the roots of the polynomial $Q(f) - f = 0$. Since f is continuous, it must be constant.

Assume now that the degree of Q is 1 and f is not constant. Since $f(x + y) = f(y + x)$, $P(u, v)$ is symmetric in u and v and must be of the form $a(u + v) + buv + c$. Setting $y = 0$ yields

$$f(x) = P(f(x), f(0)) = a(f(x) + f(0)) + bf(0)f(x) + c,$$

so $f(x)(1 - a - bf(0)) = af(0) + c$. Since f is not constant, $1 - a - bf(0) = 0 = af(0) + c$.

If $b = 0$, then $a = 1$ and $P(u, v) = u + v + c$. Hence $f(x + y) = f(x) + f(y) + c$, and so $f(0) = 2f(0) + c$ and $c = -f(0)$. Setting $g(x) = f(x) - f(0)$ yields $g(x + y) = g(x) + g(y)$ so that $g(x) = ax$ and $f(x) = ax - c$.

If $b \neq 0$, then $f(0) = (1 - a)/b = -c/a$, so $c = (a^2 - a)/b$. Hence $f(x + y) = a(f(x) + f(y)) + bf(x)f(y) + (a^2 - a)/b$, which yields

$$bf(x + y) + a = ab(f(x) + f(y)) + b^2 f(x)f(y) + a^2 = (bf(x) + a)(bf(y) + a).$$

Setting $g(x) = bf(x) + a$, we get $g(x + y) = g(x)g(y)$, and hence $g(x) = d^x$ for some $d > 0$. Thus $f(x) = (d^x - a)/b$.

Solved also by J. H. Lindsey II, A. Nijenhuis, and the proposer.