An AM–GM Variation

10672 [1998, 559]. *Proposed by V. Anil Kumar, Kerala Agricultural University, Tavanur, Kerala, India.* Let $p_1, p_2, \ldots, p_m$ be positive real numbers summing to 1, and assume that $a_{i,j} > 0$ for $1 \le i \le m$ and $1 \le j \le n$. Prove that

$$\sqrt[n]{\prod_{j=1}^{n}\left(\sum_{l=1}^{m} p_l a_{l,j} \prod_{i \ne l}\left(\sum_{k=1}^{n} a_{i,k}\right)\right)} \le \frac{1}{n}\prod_{i=1}^{m}\left(\sum_{k=1}^{n} a_{i,k}\right).$$

Solution by John H. Lindsey II, Fort Meyers, FL. With $x_j = \sum_{l=1}^{m} p_l a_{l,j} \prod_{i \ne l}(\sum_{k=1}^{n} a_{i,k})$, the left-hand side is the geometric mean of $x_1, \ldots, x_m$ and hence is less than or equal to the arithmetic mean of $x_1, \ldots, x_m$, which is

$$\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{l=1}^{m} p_l a_{l,j} \prod_{i \ne l}\left(\sum_{k=1}^{n} a_{i,k}\right)\right) = \frac{1}{n}\sum_{l=1}^{m} p_l \left(\sum_{j=1}^{n} a_{l,j} \prod_{i \ne l}\left(\sum_{k=1}^{n} a_{i,k}\right)\right)$$
$$= \frac{1}{n}\sum_{l=1}^{m} p_l \prod_{i=1}^{m}\left(\sum_{k=1}^{n} a_{i,k}\right) = \frac{1}{n}\prod_{i=1}^{m}\left(\sum_{k=1}^{n} a_{i,k}\right).$$

Solved also by S. Amighibech (France), R. J. Chapman (U. K.), Q. H. Darwish (Oman), W. Janous (Austria), B. Kalantari, S. S. Kim (Korea), M. S. Klamkin (Canada), R. Martin (U. K.), A. Nijenhuis, C. R. Pranesachar (India), H.-J. Seiffert (Germany), S. M. Soltuz (Romania), S.-E. Takahasi (Japan), T. V. Trif (Romania), GCHQ Problems Group (U. K.), and the proposer.

Functions with a Polynomial Addition Formula

10675 [1998, 560]. *Proposed by Harry Tamvakis, University of Pennsylvania, Philadelphia, PA.* Find every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that some polynomial $P(x, y) \in \mathbb{R}[x, y]$ satisfies $f(x + y) = P\big(f(x), f(y)\big)$ for every $x, y \in \mathbb{R}$.

Solution by GCHQ Problems Group, Cheltenham, U. K. The function f can take one of two forms:

(i) $f(x) = ax - c$ using $P(u, v) = u + v + c$, including the special case of constant f when $a = 0$; and

(ii) $f(x) = (d^x - a)/b$ using $P(u, v) = a(u + v) + buv + (a^2 - a)/b$.

When $y = 0$, we get $f(x) = P(f(x), f(0)) = Q(f(x))$ for some polynomial Q. If the degree of Q is more than 1, then the value of f is restricted to the roots of the polynomial $Q(f) - f = 0$. Since f is continuous, it must be constant.

Assume now that the degree of Q is 1 and f is not constant. Since $f(x+y) = f(y+x)$, $P(u, v)$ is symmetric in u and v and must be of the form $a(u + v) + buv + c$. Setting $y = 0$ yields

$$f(x) = P(f(x), f(0)) = a(f(x) + f(0)) + bf(0)f(x) + c,$$

so $f(x)(1-a-bf(0)) = af(0)+c$. Since f is not constant, $1-a-bf(0) = 0 = af(0)+c$.

If $b = 0$, then $a = 1$ and $P(u, v) = u + v + c$. Hence $f(x + y) = f(x) + f(y) + c$, and so $f(0) = 2f(0) + c$ and $c = -f(0)$. Setting $g(x) = f(x) - f(0)$ yields $g(x + y) = g(x) + g(y)$ so that $g(x) = ax$ and $f(x) = ax - c$.

If $b \ne 0$, then $f(0) = (1 - a)/b = -c/a$, so $c = (a^2 - a)/b$. Hence $f(x + y) = a(f(x) + f(y)) + bf(x)f(y) + (a^2 - a)/b$, which yields

$$bf(x + y) + a = ab(f(x) + f(y)) + b^2 f(x)f(y) + a^2 = (bf(x) + a)(bf(y) + a).$$

Setting $g(x) = bf(x) + a$, we get $g(x + y) = g(x)g(y)$, and hence $g(x) = d^x$ for some $d > 0$. Thus $f(x) = (d^x - a)/b$.

Solved also by J. H. Lindsey II, A. Nijenhuis, and the proposer.