REVIVALS

An Unsettled Inequality

10337 [1993, 798; 1995, 659]. *Proposed by Horst Alzer, Waldbröl, Germany.* Let $n \geq 1$ be an integer. Let $x_1, \ldots, x_n$ be real numbers with $x_i \in (0, 1/2]$. Consider the statement

$$\prod_{i=1}^{n} \frac{x_i}{1-x_i} \leq \frac{\sum_{i=1}^{n} x_i^n}{\sum_{i=1}^{n}(1-x_i)^n}. \qquad (\mathbf{F}_n)$$

(**a**) Prove $\mathbf{F}_n$ for $n \leq 3$.
(**b**) Show that $\mathbf{F}_n$ is false for $n \geq 6$.
(**c**)* What about $\mathbf{F}_4$ and $\mathbf{F}_5$?

Solution of part (**c**)* *by M. J. Pelling, London, England.* We show that $\mathbf{F}_4$ is true, with equality if and only if $x_1 = x_2 = x_3 = x_4$.

Write w, x, y, z for x_1, x_2, x_3, x_4, and write $\bar{w}, \bar{x}, \bar{y}, \bar{z}$ for $1-w, 1-x, 1-y, 1-z$, respectively. Then $\mathbf{F}_4$ may be written in the equivalent form

$$\frac{w^4+x^4+y^4+z^4}{wxyz} \geq \frac{\bar{w}^4+\bar{x}^4+\bar{y}^4+\bar{z}^4}{\bar{w}\bar{x}\bar{y}\bar{z}}. \qquad (1)$$

Without loss of generality, suppose that $w \geq x \geq y \geq z$. Subtracting 4 from both sides of (1) and rearranging terms leads to

$$\frac{(w^2-x^2)^2}{wxyz} + \frac{(y^2-z^2)^2}{wxyz} + \frac{2(wx-yz)^2}{wxyz} \geq \frac{(\bar{w}^2-\bar{x}^2)^2}{\bar{w}\bar{x}\bar{y}\bar{z}} + \frac{(\bar{y}^2-\bar{z}^2)^2}{\bar{w}\bar{x}\bar{y}\bar{z}} + \frac{2(\bar{w}\bar{x}-\bar{y}\bar{z})^2}{\bar{w}\bar{x}\bar{y}\bar{z}}. \qquad (2)$$

By repeated use of the elementary inequality

$$p + \frac{1}{p} \geq q + \frac{1}{q} \quad \text{whenever } p \geq q \geq 1, \qquad (3)$$

we show that each term on the left of (2) is greater than or equal to the corresponding term on the right.

Since $w + x \leq 1$, we have $w - x \geq w^2 - x^2$ or $w\bar{w} \geq x\bar{x}$. With $p = w/x$ and $q = \bar{x}/\bar{w}$, we have $p \geq q \geq 1$, so

$$\frac{(w+x)^2}{wx} \geq \frac{(\bar{w}+\bar{x})^2}{\bar{w}\bar{x}} \qquad (4)$$

by (3). Since $yz \leq \bar{y}\bar{z}$ and $(w-x)^2 = (\bar{w}-\bar{x})^2$, (4) implies

$$\frac{(w^2-x^2)^2}{wxyz} = \frac{(w+x)^2}{wx}\frac{(w-x)^2}{yz} \geq \frac{(\bar{w}+\bar{x})^2}{\bar{w}\bar{x}}\frac{(\bar{w}-\bar{x})^2}{\bar{y}\bar{z}} = \frac{(\bar{w}^2-\bar{x}^2)^2}{\bar{w}\bar{x}\bar{y}\bar{z}}. \qquad (5)$$

The same reasoning proves

$$\frac{(y^2-z^2)^2}{wxyz} \geq \frac{(\bar{y}^2-\bar{z}^2)^2}{\bar{w}\bar{x}\bar{y}\bar{z}}. \qquad (6)$$

Now let $p = wx/(yz)$ and $q = \bar{y}\bar{z}/(\bar{w}\bar{x})$. Again $p \geq q \geq 1$, so (3) implies

$$\frac{2(wx-yz)^2}{wxyz} \geq \frac{2(\bar{w}\bar{x}-\bar{y}\bar{z})^2}{\bar{w}\bar{x}\bar{y}\bar{z}}. \qquad (7)$$

Adding (5), (6), and (7) yields (2).

Since equality holds in (3) only when $p = q$, we have equality in $\mathbf{F}_4$ only if $w/x = \bar{x}/\bar{w}$, $y/z = \bar{z}/\bar{y}$, and $wx/(yz) = \bar{y}\bar{z}/(\bar{w}\bar{x})$, which forces $w = x = y = z$.

Editorial comment. Pelling also contributed a lengthy proof of $\mathbf{F}_5$ and showed that equality holds in $\mathbf{F}_5$ only when $x_1 = x_2 = x_3 = x_4 = x_5$.