REVIEWS

Edited by **Harold P. Boas**

Mathematics Department, Texas A & M University, College Station, TX 77843-3368

Wavelets: A Primer. By Christian Blatter. A K Peters, 1998, x + 202 pp., $32.
Wavelets in a Box. By Charles K. Chui, Andrew K. Chan, and C. Steve Liu. Academic Press, 1998, book and CD-ROM software, $79.95.
A Primer on Wavelets for Scientists and Engineers. By James S. Walker. CRC Press, 1999, 155 pp., $39.95.
Wavelet Analysis: The Scalable Structure of Information. By Howard L. Resnikoff and Raymond O. Wells, Jr. Springer, 1998, xvi + 435 pp., $59.95.

Reviewed by **Edward Aboufadel, Matthew Boelkins, and Steven Schlicker**

In the decade since the publication of Ingrid Daubechies' seminal book **[6]**, wavelets have captured the imagination of an increasing number of people, and not just mathematicians. There is a growing enthusiasm about the subject on the part of students and many groups of professionals. Disciplines such as radiology, geology, computer science, music, and engineering provide a wide range of applications for wavelets, including signal and image processing, denoising of data, and compression and retrieval of data. Mathematicians continue to explore the area enthusiastically, as evidenced by well-attended sessions at national meetings, with papers on topics such as wavelets and dynamical systems and the path-connectivity of a wavelet space. The study of wavelets offers an intriguing mix of linear algebra, functional analysis, and applications, and it is time to let undergraduates in on the fun.

The abstract nature of much modern mathematics often leaves students and non-mathematicians asking the question, "Why does anyone care about this stuff?" Particularly in the undergraduate curriculum, the study of groups and rings, vector spaces, inner product spaces, topological spaces, and other abstract structures makes many students question the relevance of what they are learning. We mathematicians are engaged by the beauty and elegance of our subject, which often causes us to overlook the desire of most students to connect what they are studying directly to the world in which they live.

This need for context implies that part of our job as mathematics teachers is to demonstrate the power of mathematics through applications. By "applications", we do not mean ladders sliding down walls or the production of widgets, but rather *actual* uses of mathematics. True applications often require more mathematical background or sophistication than most students have, or more non-mathematical prerequisites than can be covered in class. Nonetheless, some applications that can be explained and demonstrated rely only on basic principles that are accessible to students. Wavelets are an excellent example of an application of high-level mathematics that can be presented to undergraduates.

Not long ago, as new wavelet enthusiasts, we became interested in how to present applications of wavelets to first- and second-semester linear algebra students. In reviewing the literature, we were disappointed to find few examples of papers or books written at a level accessible to typical undergraduates (or to non-mathematicians, for that matter). While nearly all the literature was in the style of Daubechies' graduate-level text, an article by Strang **[8]** showed us that the basic ideas can be reduced to linear combinations and matrix multiplication by focusing on the Haar wavelets. This idea, together with articles appearing in the popular press (**[5]** and **[7]**) and information on the FBI's interest in compressing fingerprint images (**[2]**, **[3]**, and **[4]**), served as our starting point for introducing wavelets to undergraduates.

This approach makes wavelets an appropriate topic for inclusion in a first-semester linear algebra course. By studying the FBI's fingerprint problem, our sophomore mathematics majors—many of whom intend to be high school teachers—have become excited about a new area of mathematics and have learned how to process simple two-dimensional images using the Haar wavelets. Adding a few additional topics (orthogonality and inner product spaces, which occur in a second course on linear algebra) makes wavelets appear in an even more natural and broader setting. These experiences convinced us that wavelets are significantly more accessible than most current books suggest.

As we refined our linear algebra projects on wavelets, we set up a web site **[1]** devoted to the topic of wavelets in the undergraduate curriculum. Since establishing the site in 1998, we have received many requests for more information about wavelets from students (both in mathematics and in other disciplines) and from non-mathematicians. The vast majority of people who contact us express frustration in trying to read the currently available literature on wavelets, most of which seems to be written for specialists.

Several recently published books purport, through title, advertising, or jacket notes, to be at an introductory level. Unfortunately, the word "introductory" is not well defined. In this review, we consider four such books and address their suitability for undergraduate students or non-mathematicians.

Wavelets in a Box is a great idea. In the package are a softcover copy of the book *An Introduction to Wavelets* by Charles K. Chui together with supporting software, *Wavelet Toolware*: *Software for Wavelet Training*, by C. Steve Liu and Andrew K. Chan. The intended audience for the book can be inferred from the first sentence of Section 1.1, which reads "Let $L^2(0,2\pi)$ denote the collection of all measurable functions f defined on the interval $(0,2\pi)$ with $\int_0^{2\pi}\|f(x)\|^2\,dx<\infty$." Most (probably all) undergraduates would hesitate to read any further. To comfort those daunted by this opening statement, the author offers these soothing words: "For the reader who is not familiar with the basic Lebesgue theory, the sacrifice is very minimal by assuming that f is a piecewise continuous function." By piecewise continuous the author means "...the existence of points $\{x_j\}$ in $\mathbf{R}$ with no finite accumulation points, such that $x_i<x_j$ for all j and that f is continuous on each of the open intervals (x_j,x_{j+1}) as well as the unbounded intervals $(-\infty,\min x_j)$ and $(\max x_j,\infty)$, if $\min x_j$ or $\max x_j$ exist." This is not easy reading for an undergraduate, or even an engineer, a geologist, or a radiologist.

Nevertheless, this is a nice book if one has the appropriate background. The author states that "the only prerequisite is a basic knowledge of function theory and real analysis." More realistically, what is needed is a strong background in real analysis, a little measure theory, at least enough complex analysis to understand complex exponential functions, and significant mathematical sophistication. The lack of problems and examples also indicates that the book is at a higher level than an introduction. It includes a good primer on Fourier analysis, among other things, but this is a book written by a mathematician for mathematicians.

The companion software is a Windows-based package that can be used to process signals and images with wavelets. The accompanying guide states that *Wavelet Toolware* "is designed for the reader to gain some hands-on practice in the subject of wavelets." To a certain extent, this is correct. A student can use this package to create graphs of various famous scaling functions and mother wavelets through a built-in iterative process. One-dimensional signals can be processed via one-dimensional wavelet transformations with a choice of over a dozen different wavelet families. Students can also use the Continuous Wavelet Transform and the Short Time Fourier Transform tools. Since all of these tools are pre-coded, using the program requires almost no understanding of the mathematics of wavelets. This limits the educational value of the software.

We have found that image processing is an excellent way to motivate students, so we are glad to see that *Toolware* also contains a two-dimensional wavelet transform for the processing of grayscale images. The "2D DWT" tool reads files in binary PGM format (portable graymap, a creation of Jef Poskanzer) and creates image boxes, which are visual representations of wavelet coefficients, as shown in Figure 1. The choice of binary PGM is unfortunate since it is an uncommon format, used mostly on X Windows workstations. Consequently, students cannot easily create their own images to process. (This would have been possible if the

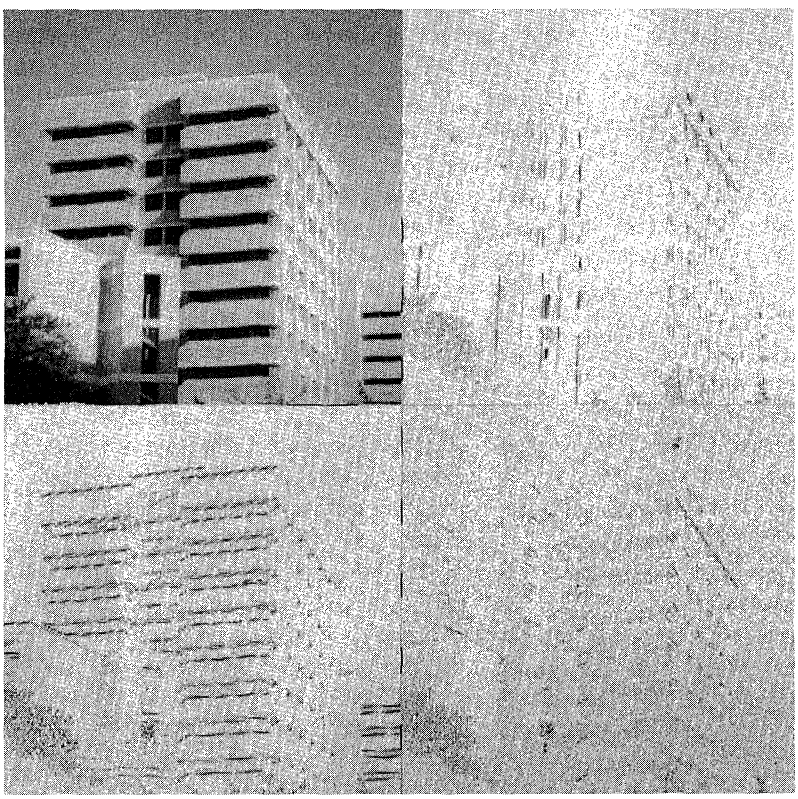

Figure 1. Image box from *Toolware*

raw PGM format had been used, since those files can be created with a text editor.) Instead, users must manipulate images from the campus of Texas A & M University, which are included in the package. This is great if you are an Aggie, but part of the fun of wavelets lies in working with one's own images. *Toolware* has a simple and straightforward user interface and would probably find its most useful role in providing in-class demonstrations.

Wavelets: A Primer is a promising title. According to the preface, the course from which the book arose was targeted towards "students of mathematics ... having the usual basic knowledge of analysis, carrying around a knapsack full of convergence theorems, but without any practical experience, say, in Fourier analysis." Actually, this slim volume belongs to the genre of wavelet books that require the reader to have a solid foundation in Fourier analysis to get past the first chapter. The author, Christian Blatter, does present some Fourier analysis in the second chapter, but the style is not very helpful to the novice.

The author remarks that this book is for students "in their senior year or first graduate year," and the latter category seems more accurate. The opening pages feature an overview of the problem of approximating functions, which is an improvement over books that begin, "A wavelet is... ." The writing is straightforward, and Blatter makes good use of figures. This text could serve as a resource for someone trying to read a book such as Chui's. Although *Wavelets: A Primer* is closer to the introductory level than other books of this type, the overall style and complete absence of problems still make this book too advanced for most undergraduate students.

Wavelet Analysis: The Scalable Structure of Information features a remarkable preface that describes the authors' history with a "mathematical engineering company" called Aware, Inc. As principal members of that company, the authors developed and implemented several ideas involving wavelets. The latter half of the book features these developments, the main purpose of *Wavelet Analysis*. The first half of the book attempts to introduce readers to the basic concepts in the study of wavelets.

The authors try to appeal to a wide audience by including a significant amount of expository material. For instance, two sections of the book are titled "Music Notation as a Metaphor for Wavelet Series" and "The Democratization of Arithmetic: Positional Notation for Numbers." An intriguing section relates wavelets to Newton's method; this could be a starting point for an undergraduate research project. A nice example to introduce students to multiresolution analysis can be found in Chapter 3, where the authors describe the "multiresolution representation for a number."

We must caution the reader that despite these fine attributes, this is an advanced work; the authors' expectations of the reader are even higher than those of Blatter and Chui. A footnote early on demonstrates that *Wavelet Analysis* is yet another book belonging to the collection of Fourier-dependent texts: the authors assume that the reader has a complete understanding of the terms "generalized Fourier series," "basis functions," and "variable compact support," which are far beyond what a typical junior mathematics major is likely to have mastered. Furthermore, in the first chapter in the wavelet theory part of the book, the authors venture off into Lie groups and their connection to wavelet matrices. Compounding the difficult nature of the text, there are no problems in the book for students to solve.

To be fair, the book does not claim to be an introduction, but rather states that "this text [is] for upper-level undergraduates and beginning graduate students" and promises to relate wavelets to "previously known methods in mathematics and engineering." Despite the expository nature of some of the book, the prospective reader should look elsewhere for an introduction to wavelets.

A Primer on Wavelets and their Scientific Applications comes closer than the other books under review to meeting the ideal of a true introductory text. James S. Walker, the author, recognizes "a real need for a simple introduction, a *primer*, which uses only elementary linear algebra and a smidgen of calculus to explain the underlying ideas behind wavelet analysis, and devotes the majority of its pages to explaining how these underlying ideas can be applied to solve significant problems in audio and image processing and in biology and medicine." He begins with an introduction to the Haar wavelets, using only a minimal amount of linear algebra, and uses them to introduce many basic ideas—from averaging and differencing to multiresolution analysis—through concrete examples. The chapter concludes with applications of the Haar wavelets to compressing and denoising audio signals. Subsequent chapters introduce the reader to the Daubechies wavelets, two-dimensional wavelet transforms, the Discrete Fourier Transform (DFT), and wavelet packets. Many applications of wavelets are presented, including compression and denoising of images, edge recognition and enhancement, image recognition, and speech analysis.

There is a lot to like in *A Primer on Wavelets*. With the exception of the chapter on frequency analysis, where the DFT is discussed, and subsequent related material, the only mathematical background necessary is some linear algebra, specifically dot products and matrix operations. The best part of the book is the depth and variety of applications that are discussed, with an emphasis throughout on the accuracy of the method being used.

Despite the excellent overview of applications, the book falls short of being an ideal introduction for students. Other than in the first chapter and the sections on wavelet packets, the book does not work through specific examples in detail. In addition, despite the jacket's claim that "throughout the text are numerous suggestions for computer experiments and exercises," there are *no* problems for the reader to work.

Walker has written an impressive, freely available program FAWAV [**9**] to demonstrate wavelets in action. This Windows-based package, which "*requires no programming to use*" (author's emphasis), enables the user to process one- and two-dimensional signals and images. The program even contains a basic audio editor for clipping portions of sound files to study.

With a modest amount of experimentation and on-line help, the user is soon able to use FAWAV to plot functions, load two-dimensional grayscale images in a broad range of formats (including PGM), and manipulate audio clips. Through the transform feature, the wavelet enthusiast can choose from a variety of Haar, Daubechies, and Coifman transforms to see the transform and inverse transform of a signal in a step-by-step fashion. Via wavelet series, one can experiment with various levels of thresholding and see the end results of compression and denoising. It is here that the software may be most valuable to students, for through trial and error they can see how well (or poorly) information can be retrieved following compression or denoising (see Figure 2). The program also includes wavelet packets, Fourier transforms, and a varied collection of tools for detailed study of the effectiveness and accuracy of signal compression. FAWAV offers great potential for in-class demonstrations of many applications of wavelets.

Walker's *Primer* is filled with many figures of FAWAV output that illustrate key ideas. These images, like his text, do much to show the powerful results of wavelet

Figure 2. FAWAV output: a noisy *Lena* (Gr 1), a denoised *Lena* (Gr 2), significance map from thresholded transform used in denoising (Gr 3), and the original *Lena* (Gr 4).

analysis. While this convinces the reader that wavelets work, it often leaves one wondering *how* they work. Like *Toolware*, all of FAWAV is pre-coded, so the algorithms are inaccessible. This limits the program's value for introducing the ideas behind the output to students. In addition, although the author claims that the software is designed to enable the reader to "duplicate all of the applications described in this primer", we were unable to do so in a few cases. A collection of sample exercises to introduce the user to FAWAV would be a valuable improvement; the omission of such exercises and supplementary by-hand problems is the work's greatest shortcoming.

Easily the most accessible text among those under consideration, *A Primer on Wavelets and Their Scientific Applications* is an excellent resource book, especially for its overview of applications.

Of the several dozen books on wavelets published in the last ten years, most have endeavored to create comprehensive and rigorous presentations. With the exception of Walker's text, the books discussed in this review belong to this class. While they may have desired to write introductions, most of the authors of these books have fallen into a familiar trap: considering the most general case first (which, for wavelets, involves a treasure trove of Fourier transform theory) and using an occasional specific case as an example. This is fine for a rigorous, abstract reference book, but it is deadly when trying to *introduce* a topic to a broad audience, particularly one including undergraduate students. And though Walker writes for a broader audience of scientists, his book is not ideally suited to study by undergraduates or novices.

The number of requests received at our web site leads us to believe that there remains a real need for a book on the topic that is written at a truly introductory level. Such a text would be geared to individuals who need an entry point to the more technical books and papers, would provide an appropriate amount of detail (via linear algebra) as to how wavelets work, and would appeal to undergraduate students and non-mathematicians. With its beauty, power, and accessibility, the subject deserves a presentation that further widens the growing collection of wavelet enthusiasts.

REFERENCES

1. Edward Aboufadel and Steven Schlicker, Discovering Wavelets website, `http://www.gvsu.edu/mathstat/wavelets.htm`.
2. Jonathan N. Bradley, Christopher M. Brislawn, and Tom Hopper, The FBI wavelet/scalar quantization standard for grey-scale fingerprint image compression, in *Visual Info. Process. II*, volume 1961 of Proc. SPIE, Orlando, FL, 1993, pp. 293–304.
3. Christopher M. Brislawn, Fingerprints go digital, *Notices Amer. Math. Soc.* **42** (1995) 1278–1283.
4. Barry Cipra, Wavelet applications come to the fore, *SIAM News* **26-7**, no. 7, November 1993, also available online at `http://www.siam.org/siamnews/mtc/mtc1193.htm`.
5. Steven Courtney, Information age multiplies uses of mathematics formulas for the future, *The Hartford Courant*, October 21, 1993, p. E1.
6. Ingrid Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, 1992.
7. Peter Schröder, Wavelet image compression: Beating the bandwidth bottleneck, *Wired*, May 1995, p. 78.
8. Gilbert Strang, Wavelet transforms versus Fourier transforms, *Bull. Amer. Math. Soc.* **28** (1993) 288–305.
9. James S. Walker, FAWAV wavelets software, available online at `http://www.crcpress.com/edp/download/fawav/fawav.htm`.

Grand Valley State University, Allendale, MI 49401
aboufade@gvsu.edu, boelkinm@gvsu.edu, schlicks@gvsu.edu

Poincaré and the Three Body Problem. By June Barrow-Green. American Mathematical Society, 1997, 272 pp., $39.

Reviewed by **Daniel Henry Gottlieb**

In a work of impressive scholarship, the author takes us through the history of the n-body problem from Newton to the present. The center of her story is the prize competition in honor of the 60th birthday of King Oscar II of Sweden in 1889. With royal patronage, with the most prestigious mathematicians as judges, and with the momentous mathematical problem of Civilization as a topic, it had captured the attention of the mathematical world. And the winner was... Poincaré... with a manuscript that had a major error!

The paper was due to be published on the King's birthday a few weeks hence, when Poincaré himself discovered the false result. The difficulty of his position was enormous. An error in a paper so highly honored not only would be a great personal embarrassment, but would damage the reputations of the judges and the organizers of the competition as well as ruin the King's birthday.