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NOTES

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Suppose that water towers like those shown in Figure 1 are initially filled with water and have the same volume, height, and cross-sectional outlet area. Which one empties first? This problem arises naturally when designing water-supplying tanks or funnels.

We find a formula expressing the emptying time as a function of the volume of liquid and its initial height. We compute the emptying time for several specific tank shapes, in particular, for those shown in Figure 1. We also address the question whether there exists a tank with a given volume and height for which the emptying time is minimal.

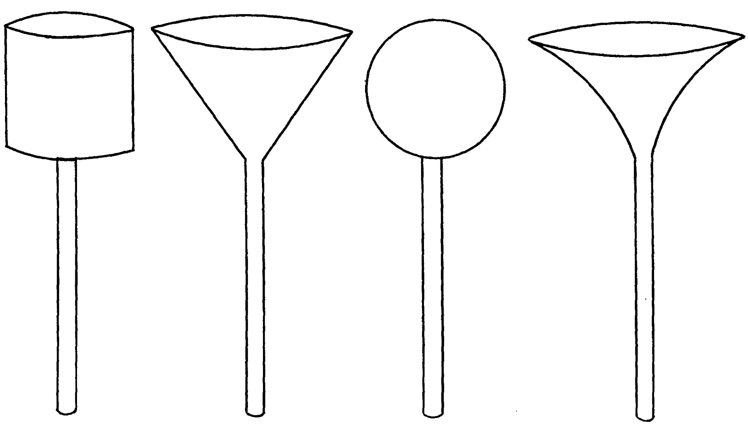

Figure 1

1. Mathematical Model. Suppose a tank of volume V and height H is initially filled with an incompressible liquid. A small (but not microscopic) hole with cross-sectional area S is made in the bottom of the tank. Let $A(h)$, $0 \leq h \leq H$, be the area of the tank cross-section at height h. We assume that the function $A(h)$ is continuous.

Let h be the height of the liquid in the tank at a time t. Let Δh be the height drop during a small amount of time Δt that elapses from the moment t. Then the volume decrease ΔV approximately equals $A(h)\Delta h$. As we show later, the velocity of the outgoing flow is a function of liquid height: $v = v(h)$. Hence, the volume of the liquid leaving the tank during the time Δt is approximately equal to $Sv(h)\Delta t$. Thus, $A(h)\Delta h \simeq -Sv(h)\Delta t$. Letting $\Delta t \to 0$, we obtain the differential equation

$$h' = -\frac{S}{A(h)}v(h). \tag{1}$$

To solve it, the function $v(h)$ must be specified.

2. Torricelli's Law. In 1640, E. Torricelli found that

$$v(h) = \sqrt{2gh}, \tag{2}$$

where g is the acceleration due to gravity. Here is a simple argument for (2); see also [**1**] and [**2**]. Let Δm be the mass of the liquid leaving the tank during the time Δt. Then the potential energy loss $\Delta\Pi$ is approximately equal to Δmgh. The kinetic energy ΔK of the equal amount of liquid flowing out of the tank through the orifice during the time Δt is about $\Delta mv^2(h)/2$. Equating $\Delta\Pi$ and ΔK gives (2). For a careful derivation of Torricelli's formula, see [**3**, pp. 47–48 and 56–59].

In reality, due to viscosity of the liquid, its rotation, and constriction of the jet emerging from the tank, (2) is not quite accurate, especially in the case of non-horizontal outflow. Experiments show that for a circular orifice

$$v(h) = \alpha\sqrt{gh}, \tag{3}$$

where the constant α depends on the physical properties of the liquid [**3**, pp. 47–48]. For example, the approximate value of the coefficient α for water is 0.84.

If liquid were oozing from the tank at the constant initial rate $v_0 = v(H) = \alpha\sqrt{gH}$, then the emptying time T^* would be

$$T^* = \frac{V}{Sv_0} = \frac{V}{S\alpha\sqrt{gH}}. \tag{4}$$

However, according to Torricelli's law, the efflux rate is decreasing with the decrease of height. Therefore, the emptying time T is

$$T = kT^* = k\frac{V}{S\alpha\sqrt{gH}}, \tag{5}$$

where $k > 1$. In general, the coefficient k depends on the height H and the shape of the tank. We show, however, that for many practically important tank forms, the coefficient k is an absolute constant.

3. Emptying Time. With (3) taken into account, the differential equation (1) takes on the following form:

$$h' = -S\alpha\sqrt{g}\,\frac{\sqrt{h}}{A(h)}. \tag{6}$$

Some properties of this equation are discussed in [**1**] and [**2**]. A classroom demonstration based on this equation for a cylindrical container is described in [**4**].

The solution $h = h(t)$ of (6) satisfying the initial condition $h(0) = H$ is given implicitly by

$$\int_h^H \frac{A(u)}{\sqrt{u}}du = S\alpha\sqrt{g}\,t. \tag{7}$$

Since $h(T) = 0$, we obtain from (7) that

$$T = \frac{1}{S\alpha\sqrt{g}}\int_0^H \frac{A(u)}{\sqrt{u}}du. \tag{8}$$

This formula provides a closed expression for the emptying time T.

Observing that

$$V = \int_0^H A(u)\,du \tag{9}$$

we rewrite (8) in the form (5), where

$$k = k(H) = \sqrt{H}\,\frac{\int_0^H \left(A(u)/\sqrt{u}\right)du}{\int_0^H A(u)\,du}. \tag{10}$$

We set

$$g(s) := A(s^2), \quad 0 \le s \le \sqrt{H}, \tag{11}$$

in (9) and (10) to find that

$$V = 2\int_0^{\sqrt{H}} g(s)s\,ds \quad \text{and} \quad \int_0^H \frac{A(u)}{\sqrt{u}}\,du = \int_0^{\sqrt{H}} g(s)\,ds. \tag{12}$$

This leads to the following alternative expressions for k:

$$k = \sqrt{H}\,\frac{\int_0^{\sqrt{H}} g(s)\,ds}{\int_0^{\sqrt{H}} g(s)s\,ds} = \frac{\int_0^1 g(\sqrt{H}s)\,ds}{\int_0^1 g(\sqrt{H}s)s\,ds}. \tag{13}$$

The case of a circularly symmetric tank is probably the most important. The lateral surface of such tank is obtained by rotating the graph of a nonnegative continuous function $f(h)$, $0 \le h \le H$, about the h axis. Then

$$A(h) = \pi f^2(h). \tag{14}$$

Suppose f is homogeneous of some order $\theta \ge 0$: that is, for any $\lambda > 0$ and for all admissible $h \in [0, H]$,

$$f(\lambda h) = \lambda^\theta f(h). \tag{15}$$

Then the function g defined by (11) is homogeneous of order 4θ. In view of (13), this leads to the important conclusion that in this case the coefficient k depends only on f, that is, only upon the shape of the tank.

We compute the coefficient k for a few simple and widely used tank shapes, including those in Figure 1. Formula (5) then gives the emptying time.

Cylinder. Let the tank be a right circular cylinder of height H with base radius R, where $R^2 = V/(\pi H)$; see Figure 1a. In this case, $f(h) = R$, $0 \le h \le H$, which is a homogeneous function of order 0. Then $g(s) = \pi R^2$, and therefore by (13), $k = 2$.

Cone. For the tank in the form of a right circular cone (Figure 1b) with height H and radius R, where $R^2 = 3V/(\pi H)$, we have $f(h) = \gamma h$ with $\gamma = H/R$. Then f satisfies (15) with $\theta = 1$, and it follows easily from (13) that $k = 1.2$.

Frustum of a cone. Suppose the tank has the form of a right circular frustum of a cone with lower base radius R_1 and upper base radius R_2. Then $f(h) = a + bh$, where $a = R_1$ and $b = (R_2 - R_1)/H$, and $g(s) = \pi(a + bs^2)^2$. A straightforward calculation based on (13) gives the following expression for the coefficient k:

$$k = \frac{2}{5}\cdot\frac{8R_1^2 + 4R_1R_2 + 3R_2^2}{R_1^2 + R_1R_2 + R_2^2}. \tag{16}$$

Thus, k is independent of H. For $R_1 = 0$, (16) yields the value 1.2 already obtained for the cone. In the other extreme case of the inverse cone ($R_2 = 0$), we have $k = 3.2$. For $R_1 = R_2$, (16) produces the value $k = 2$ already found earlier for the cylinder.

Spherical tanks. Let the tank be a truncated sphere of height H, which is the most popular form for aquariums. The radius R of the sphere is determined by the tank volume through the formula $V = \pi H^2(R - H/3)$. In this case, $f^2(h) = h(2R - h)$, $0 \le h \le H$. Hence, $g(s) = \pi s^2(2R - s^2)$, and by (13) we obtain after a short calculation that

$$k(H) = \frac{2}{5} \cdot \frac{10R - 3H}{3R - H}.$$

In particular, for a hemispherical tank ($H = R$), we find that $k = 1.4$ while for a complete spherical tank ($H = 2R$; see Figure 1c), we obtain $k = 1.6$.

Table 1 summarizes our results and shows the relative emptying efficiency of various tank forms. The conic shape turns out to be significantly more efficient than other natural shapes. This explains why it is so widely used for funnels. Formula (5) and Table 1 allow us to compare emptying times of tanks of various shapes with variable volume and height.

TABLE 1 "Emptying efficiencies" of different tank shapes

Tank Shape	Cone	Hemisphere	Sphere	Cylinder	Inverse cone
k	1.2	1.4	1.6	2	3.2

For physical reasons, the coefficient k is always larger than 1. Can it be less than 1.2? As shown in the next section, the answer to this question is YES!

4. Are there Tanks with the Minimal Emptying Time? Let the function that determines the shape of a circularly symmetric tank be

$$f(h) = Ch^{\mu}, \quad 0 \le h \le H, \tag{17}$$

with some constants $\mu \ge 0$ and $C > 0$. Given μ, the value of C can be found from the relation $V = \pi C^2 H^{2\mu+1}/(2\mu + 1)$. The function (17) is homogeneous of order $\theta = \mu$. Then $g(s) = \pi C^2 s^{4\mu}$, and using (13) we obtain easily that

$$k = \frac{4\mu + 2}{4\mu + 1}. \tag{18}$$

For $\mu = 0$ and $\mu = 1$, we recover the values of $k = 2$ for the cylinder and $k = 1.2$ for the cone, respectively. For $\mu = 2$, we have $k = 10/9$, which means that, for the parabolic tank shown in Figure 1d, the emptying time is more than 7% smaller than for the corresponding conic tank in Figure 1b.

It follows from (18) that, for power functions (17) with large μ, the coefficient k can be as close to 1 as we wish. Therefore, the emptying time can be arbitrarily close to its theoretical minimum (4). We show, however, that the minimum is not attained. This means that among all tanks with a given volume and height none has the minimal emptying time. Our argument also provides a mathematical proof that $k > 1$.

Consider tanks with a given volume V and height H. We continue to assume that the cross-sectional area $A(h)$ at height h is a continuous function of h.

According to (8) and (12), we are dealing with the extremal problem

$$\mathscr{T}(g) := \int_0^a g(s)\, ds \longrightarrow \min$$

subject to the constraint $\int_0^a g(s) s\, ds = b$, where $a = \sqrt{H}$, $b = V$, and g belongs to the class G of nonnegative continuous functions on $[0, a]$. Consider a similar problem

$$\mathscr{T}(\nu) := \nu([0, a]) \longrightarrow \min, \quad \int_0^a s\, d\nu(s) = b \tag{19}$$

on the larger class N of nonnegative finite Borel measures ν on $[0, a]$. For every $\nu \in N$, we have

$$\mathscr{T}(\nu) = \int_0^a d\nu(s) \geq \frac{1}{a}\int_0^a s\, d\nu(s) = \frac{b}{a} = \mathscr{T}\left(\frac{b}{a}\delta_a\right),$$

where δ_a is the Dirac measure at a. Therefore, the measure $\nu^* := b\delta_a/a$ is a minimizer of the functional $\mathscr{T}$ on the set N, and the minimum value of $\mathscr{T}$ on N is equal to b/a. Taking a clue from (13), we find that the corresponding minimal value of k is

$$k^* = a\frac{\int_0^a d\nu^*(s)}{\int_0^a s\, d\nu^*(s)} = 1.$$

If for some $\nu \in N$ we have $\mathscr{T}(\nu) = b/a$, then $\int_0^a (a - s) d\nu(s) = 0$, whence it follows that ν is proportional to the Dirac measure at a. Therefore, (19) ensures that $\nu = \nu^*$. Thus, the minimizer ν^* is unique. This implies that the infimum of the functional $\mathscr{T}$ on the set G is not attained. However, there are sequences of functions in G for which the corresponding values of the functional $\mathscr{T}$ converge to b/a. One of them is a sequence of functions g_n that are related via (11) and (14) to functions (17) with a sequence μ_n tending to infinity.

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