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A *path* is a finite sequence of ± 1 's with a graphical representation as a sequence of contiguous steps of slope $+1$ (upsteps) and -1 (downsteps). For example, the path $w = (1, -1, -1, 1, -1, 1, 1, -1)$ is pictured in Figure 1.

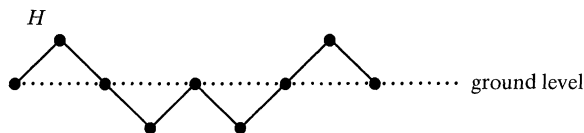


Figure 1

Let \mathcal{P}_n denote the set of $\binom{2n}{n}$ paths consisting of n upsteps and n downsteps. Each path in \mathcal{P}_n starts and terminates at “ground level” as in Figure 1. There is a well known parameter (statistic) on \mathcal{P}_n that we will call *northcnt* (to suggest a count north of a baseline). For $w \in \mathcal{P}_n$, $\text{northcnt}(w)$ is the number of w 's n upsteps that lie above ground level. Thus $\text{northcnt} = 2$ in Figure 1, and as w ranges over \mathcal{P}_n northcnt has possible values 0 through n . The paths for which $\text{northcnt} = n$ —that is, the paths that lie entirely at or above ground level—we call *Catalan* paths. Dually, we call the paths with $\text{northcnt} = 0$ *inverted Catalan* paths: reflection in ground level gives a bijection between the two classes. It is a famous fact that exactly $1/(n + 1)$ of the paths in \mathcal{P}_n are Catalan: they are counted by the Catalan number $\frac{1}{n + 1} \binom{2n}{n}$. A combinatorially satisfying way to see this is via the Chung-Feller Theorem, which asserts that the parameter northcnt is in fact *uniformly* distributed on $[0, n]$. This partitions \mathcal{P}_n into $n + 1$ equal-size classes, one of which consists of the Catalan paths. For combinatorial proofs of the Chung-Feller Theorem, see [1], [2], [3], or [4].

Curiously, there is another parameter on \mathcal{P}_n , *westcnt*, that serves the same purpose: it is also uniformly distributed on $[0, n]$ and it has a constant value on the set of inverted Catalan paths. To define $\text{westcnt}(w)$, let H denote the highest point of w , taking the leftmost one if there is more than one highest point as in Figure 1. Then $\text{westcnt}(w)$ is the number of w 's n upsteps that lie to the left (west) of H . Thus the path in Figure 1 has $\text{westcnt} = 1$, and $\text{westcnt} = 0$ precisely for the inverted Catalan paths. The parameter westcnt is implicit in [5].

One could show directly that westcnt is uniformly distributed on $[0, n]$. This is essentially done in [5], modulo translation from bracket sequences to lattice paths. But that still leaves open the question, why? Can one “explain” why northcnt and westcnt are equidistributed? A satisfactory answer would consist of a “nice” bijection $\phi : \mathcal{P}_n \rightarrow \mathcal{P}_n$ such that $\text{westcnt}(w) = \text{northcnt}(\phi(w))$ for all $w \in \mathcal{P}_n$. Here we give a simple such bijection.

To define ϕ , first observe that every path in \mathcal{P}_n can be uniquely decomposed as in Figure 2 where the C_i and D_i are inverted Catalan paths (possibly empty), lying

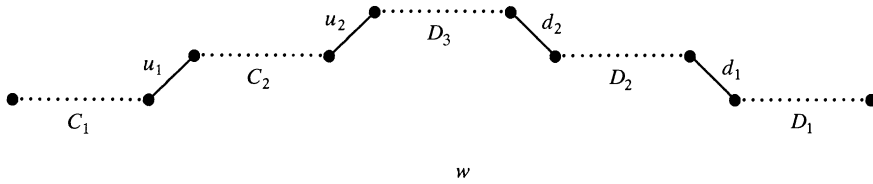


Figure 2

below the dotted segments. Each u_i is an upstep and each d_i is a downstep. There will be k C 's and $k + 1$ D 's for some $k \geq 0$; in the illustration, $k = 2$. To see uniqueness, imagine the space above ground level divided into horizontal strips as indicated by the dotted lines (extended) in Figure 2. Then u_i, d_i are respectively the leftmost upstep and rightmost downstep in the i th strip above ground level.

The path $\phi(w)$ is given by flipping over each C_i path so it becomes a Catalan path C'_i and then rearranging components as in Figure 3. Note that since H (the

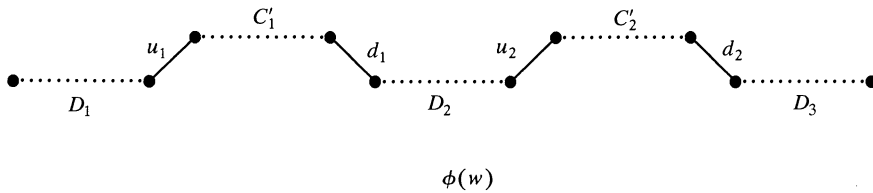


Figure 3

leftmost high point) is the northeast tip of of u_k (of u_2 in Figure 2)

$$\text{westcnt}(w) = \# u\text{'s} + \text{total } \# \text{ upsteps in the } C_i.$$

Also,

$$\text{northcnt}(\phi(w)) = \# u\text{'s} + \text{total } \# \text{ upsteps in the } C'_i.$$

However, for each i , $\#$ upsteps in $C_i = \#$ downsteps in $C_i = \#$ upsteps in C'_i , and hence $\text{westcnt}(w) = \text{northcnt}(\phi(w))$, as desired.

Finally, to show ϕ is a bijection, we must check reversibility: can the u_i, d_i, C'_i, D_i as in Figure 3 be retrieved uniquely from each path in \mathcal{P}_n ? Yes: consider the first horizontal strip above ground level. Traversing this strip left to right, upsteps and downsteps are encountered alternately. These determine the u_i and d_i (if any). The connecting paths (possibly empty) determine the C'_i and D_i in order. We are done.

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