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# **Determinants of Commuting-Block Matrices**

#### **Istvan Kovacs, Daniel S. Silver, and Susan G. Williams**

Let  $\mathcal R$  be a commutative ring, and let Mat<sub>n</sub>( $\mathcal R$ ) denote the ring of  $n \times n$  matrices over R. We can regard a  $k \times k$  matrix  $M = (A^{(i,j)})$  over Mat<sub>u</sub>(R) as a block *matrix*, a matrix that has been partitioned into  $k^2$  submatrices (blocks) over  $\mathcal{R}$ , each of size  $n \times n$ . When M is regarded in this way, we denote its determinant in  $\mathcal{R}$  by |M|. We use the symbol  $D(M)$  for the determinant of M viewed as a  $k \times k$ matrix over Mat<sub>n</sub>( $\mathcal{R}$ ). It is important to realize that  $D(M)$  is an  $n \times n$  matrix.

**Theorem 1.** Let  $\mathcal{R}$  be a commutative ring. Assume that M is a  $k \times k$  block matrix of blocks  $A^{(i,j)} \in Mat_n(\mathcal{R})$  that commute pairwise. Then

$$
|M| = |D(M)| = \left| \sum_{\pi \in S_k} (\text{sgn } \pi) A^{(1, \pi(1))} A^{(2, \pi(2))} \cdots A^{(k, \pi(k))} \right|.
$$
 (1)

Here  $S_k$  is the symmetric group on k symbols; the summation is the usual one that appears in the definition of determinant. Theorem 1 is well known in the case  $k = 2$ ; the proof is often left as an exercise in linear algebra texts; see [4, p. 164]. The general result is implicit in **[3],** but it is not widely known. We present a short, elementary proof using mathematical induction on  $k$ . We sketch a second proof when the ring  $\mathscr R$  has no zero divisors, a proof that is based on [3] and avoids induction by using the fact that commuting matrices over an algebraically closed field can be simultaneously triangularized.

*Proof:* We use induction on k. The case  $k = 1$  is evident. We suppose that (1) is true for  $k - 1$  and then prove it for k. Observe that the following matrix equation holds:

$$
\begin{pmatrix} I & O & \cdots & O \\ -A^{(2,1)} & I & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ -A^{(k,1)} & O & \cdots & I \end{pmatrix} \begin{pmatrix} I & O & \cdots & O \\ O & A^{(1,1)} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A^{(1,1)} \end{pmatrix} M = \begin{pmatrix} A^{(1,1)} & * & * & * \\ O & & & \\ \vdots & & N & \\ O & & & \end{pmatrix},
$$

where N is a  $(k - 1) \times (k - 1)$  matrix. For the sake of notation, we write this as

$$
PQM = R,\t\t(2)
$$

where the symbols are defined appropriately. By the multiplicative property of determinants we have  $D(PQM) = D(P)D(Q)D(M) = (A^{(1,1)})^{k-1}D(M)$  and  $D(R) = A^{(1, 1)}D(N)$ . Hence we have  $(A^{(1, 1)})^{k-1}D(M) = A^{(1, 1)}D(N)$ . Take the determinant of both sides of the last equation. Using  $|D(N)| = |N|$ , a consequence of the induction hypothesis, together with (2), we find

$$
\left| A^{(1,1)} \right|^{k-1} \left| D(M) \right| = \left| A^{(1,1)} \right| \left| D(N) \right| = \left| A^{(1,1)} \right| \left| N \right|
$$
  
= 
$$
\left| R \right| = \left| P \right| \left| Q \right| \left| M \right| = \left| A^{(1,1)} \right|^{k-1} \left| M \right|.
$$

If  $|A^{(1,1)}|$  is neither zero nor a zero divisor, then we can divide the sides by  $|A^{(1,1)}|^{k-1}$  to get (1). For the general case, we embed  $\mathcal{R}$  in the polynomial ring

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 $\mathcal{R}[z]$ , where z is an indeterminant, and replace  $A^{(1,1)}$  by the matrix  $zI + A^{(1,1)}$ . Since the determinant of  $zI + A^{(1, 1)}$  is a monic polynomial of degree *n*, and hence is neither zero nor a zero divisor, (1) holds again. Substituting  $z = 0$  (equivalently, equating constant terms of both sides) yields the desired result.

We sketch an alternative proof of Theorem 1 when  $\mathcal R$  has no zero divisors, a proof suggested to us by the referee. It is based on ideas of [3]; see also [1]. If  $\mathcal{R}$  is a commutative ring with no zero divisors, then we can embed it in its quotient field and then pass to the algebraic closure F. We now regard the blocks  $A^{(i,j)}$  as operators on the vector space  $F<sup>n</sup>$ , and M as an operator on the tensor product  $V = F^n \otimes F^k$ . Since the blocks  $A^{(i,j)}$  commute pairwise, there exists a basis for  $F^n$ with respect to which each  $A^{(i,j)}$  is upper triangular; see [2, p. 108]. We form the tensor product of such a basis with the standard one for  $F<sup>k</sup>$ , thereby constructing a new basis for  $V$ . The change of basis has the effect on  $M$  of simultaneously triangularizing each block. Thus it suffices to assume that each block  $A^{(i,j)}$  is upper triangular.

The matrix M is permutation-similar to a  $n \times n$  block matrix  $\tilde{M} = (\tilde{A}_{p,q})$  such that  $\tilde{A}_{p,q} = (A_{p,q}^{(i,j)})$  is an  $k \times k$  matrix consisting of the p, q-entries of the  $A^{(i,j)}$ . that  $A_{p,q} = (A_p^{(k)}, A_p^{(l)})$  is an  $k \times k$  matrix consisting of the p, q-entries of the  $A_k^{(k)},$ .<br>Since each  $A_j^{(l)}$  is upper triangular,  $\tilde{A}_{p,q} = 0$  whenever  $p > q$ . Hence  $|\tilde{M}| = |\tilde{A}_{1,1}| \cdots |\tilde{A}_{n,n}| = \prod_{r=1}^n \sum_{\pi \in S$ Since each  $A^{(k,j)}$  is upper triangular,  $A_{p,q} = 0$  whenever  $p > q$ . Hence  $|M| = |\tilde{A}_{1,1}| \cdots |\tilde{A}_{n,n}| = \prod_{r=1}^{n} \sum_{\pi \in S_k} (\text{sgn } \pi) A^{(1,\pi(1))}_{r,r} \cdots A^{(k,\pi(k))}_{r,r}$ . Since each  $A^{(i,j)}$  is upper triangular, the last product is  $|A_{1,1}| \cdots |A_{n,n}| = \prod_{r=1}^{n} \sum_{\pi \in S_k} (\text{sgn }\pi) A_{r,r}^{(r,r)} \cdots A_{r,r}^{(r,n)}$ . Since each  $A^{(r,r)}$ <br>triangular, the last product is equal to  $\prod_{r=1}^{n} \sum_{\pi \in S_k} (\text{sgn }\pi) (A^{(1,\pi(1))} \cdots A^{(1,\pi(n))})$ .<br>But this is equal to  $|\sum_{\pi \in S_k} (\$ 

The second proof shows that the commutativity hypotheses of Theorem 1 can be replaced by the weaker condition that the blocks can be simultaneously triangularized. However, some hypothesis about the blocks is certainly needed for the conclusion of the theorem to hold, as the reader can see by considering the matrix

$$
M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

We conclude by describing a class of block matrices that satisfy the commutativity hypothesis of Theorem 1. Matrices of this type arose in [S], and were the original motivation for this investigation. Let  $p^{(i,j)}(t)$  be polynomials,  $1 \le i, j \le k$ , and let N be an  $n \times n$  matrix. All coefficients are in  $\mathcal{R}$ , which can be taken to be the field of complex numbers, if the reader desires. Since the matrices  $p^{(i,j)}(N)$ commute pairwise, the block matrix

$$
M = \begin{pmatrix} p^{(1,1)}(N) & \cdots & p^{(1,k)}(N) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(N) & \cdots & p^{(k,k)}(N) \end{pmatrix}
$$

satisfies the hypothesis of Theorem 1. In fact, using the theorem we can say something about the determinant of M. Let  $p(t)$  be the determinant of

$$
\begin{pmatrix}p^{(1,1)}(t) & \cdots & p^{(1,k)}(t) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(t) & \cdots & p^{(k,k)}(t)\end{pmatrix},
$$

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and let  $\zeta_1, \ldots, \zeta_n$  be the (not necessarily distinct) eigenvalues of N. We leave the proof of the following assertion as an exercise:

$$
|M|=\prod_{r=1}^n p(\zeta_r).
$$

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## **Mixtilinear Incircles**

#### **Paul Yiu**

L. Bankoff [1] has coined the term *mixtilinear incircles* of a triangle for the three circles each tangent to two sides and to the circumcircle internally. Consider a triangle ABC and its mixtilinear incircle in the angle A, with center  $K<sub>A</sub>$ , and radius  $p_A$ . Bankoff has established the fundamental formula

$$
r = \rho_A \cdot \cos^2 \frac{\alpha}{2},\tag{1}
$$

where r is the inradius of the triangle, and  $\alpha$  is the magnitude of the angle at A. This formula had appeared earlier as an exercise in **[2,** p. 231. It leads to a simple construction of the mixtilinear incircle. Denote by I the incenter of triangle  $ABC$ , and let the perpendicular through I to the bisector of angle  $\Lambda$  intersect the sides  $AC$ , AB at  $Y_1$  and  $Z_1$ , respectively. The perpendiculars at these points to their respective sides intersect again on the angle bisector, at the mixtilinear incenter  $K_A$ . The circle with center  $K_A$ , passing through  $Y_1$  (and  $Z_1$ ), is the mixtilinear incircle in angle  $A$ ; see Figure 1.

In this note, we demonstrate the usefulness of the notion of barycentric coordinates in discovering remarkable geometric properties relating to the mixtilinear incircles of a triangle. To keep the note self-contained, we refrain from using (I), except for the remarks at the end.

Denote by  $A'$  the point of contact of the mixtilinear incircle in angle  $A$  with the circumcircle. For convenience, we denote  $K_A$  by K, and  $\rho_A$  by  $\rho$  when there is no danger of confusion; see Figure 2. The center  $K$  lies on the bisector of angle  $A$ , and  $AK: KI = \rho: -(\rho - r)$ . In terms of barycentric coordinates,

$$
K = \frac{1}{r} [-(\rho - r)A + \rho I]. \tag{2}
$$