Determinants of Commuting-Block Matrices

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Let $\mathscr{R}$ be a commutative ring, and let $\mathrm{Mat}_n(\mathscr{R})$ denote the ring of $n \times n$ matrices over $\mathscr{R}$. We can regard a $k \times k$ matrix $M = (A^{(i,j)})$ over $\mathrm{Mat}_n(\mathscr{R})$ as a *block matrix*, a matrix that has been partitioned into k^2 submatrices (*blocks*) over $\mathscr{R}$, each of size $n \times n$. When M is regarded in this way, we denote its determinant in $\mathscr{R}$ by $|M|$. We use the symbol $D(M)$ for the determinant of M viewed as a $k \times k$ matrix over $\mathrm{Mat}_n(\mathscr{R})$. It is important to realize that $D(M)$ is an $n \times n$ matrix.

Theorem 1. *Let $\mathscr{R}$ be a commutative ring. Assume that M is a $k \times k$ block matrix of blocks $A^{(i,j)} \in \mathrm{Mat}_n(\mathscr{R})$ that commute pairwise. Then*

$$|M| = |D(M)| = \left| \sum_{\pi \in S_k} (\operatorname{sgn} \pi) A^{(1,\pi(1))} A^{(2,\pi(2))} \cdots A^{(k,\pi(k))} \right|. \qquad (1)$$

Here S_k is the symmetric group on k symbols; the summation is the usual one that appears in the definition of determinant. Theorem 1 is well known in the case $k = 2$; the proof is often left as an exercise in linear algebra texts; see [**4**, p. 164]. The general result is implicit in [**3**], but it is not widely known. We present a short, elementary proof using mathematical induction on k. We sketch a second proof when the ring $\mathscr{R}$ has no zero divisors, a proof that is based on [**3**] and avoids induction by using the fact that commuting matrices over an algebraically closed field can be simultaneously triangularized.

Proof: We use induction on k. The case $k = 1$ is evident. We suppose that (1) is true for $k - 1$ and then prove it for k. Observe that the following matrix equation holds:

$$\begin{pmatrix} I & O & \cdots & O \\ -A^{(2,1)} & I & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ -A^{(k,1)} & O & \cdots & I \end{pmatrix} \begin{pmatrix} I & O & \cdots & O \\ O & A^{(1,1)} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A^{(1,1)} \end{pmatrix} M = \begin{pmatrix} A^{(1,1)} & * & * & * \\ O & & & \\ \vdots & & N & \\ O & & & \end{pmatrix},$$

where N is a $(k-1) \times (k-1)$ matrix. For the sake of notation, we write this as

$$PQM = R, \qquad (2)$$

where the symbols are defined appropriately. By the multiplicative property of determinants we have $D(PQM) = D(P)D(Q)D(M) = (A^{(1,1)})^{k-1}D(M)$ and $D(R) = A^{(1,1)}D(N)$. Hence we have $(A^{(1,1)})^{k-1}D(M) = A^{(1,1)}\ D(N)$. Take the determinant of both sides of the last equation. Using $|D(N)| = |N|$, a consequence of the induction hypothesis, together with (2), we find

$$\begin{aligned} |A^{(1,1)}|^{k-1} |D(M)| &= |A^{(1,1)}|\, |D(N)| = |A^{(1,1)}|\, |N| \\ &= |R| = |P|\, |Q|\, |M| = |A^{(1,1)}|^{k-1} |M|. \end{aligned}$$

If $|A^{(1,1)}|$ is neither zero nor a zero divisor, then we can divide the sides by $|A^{(1,1)}|^{k-1}$ to get (1). For the general case, we embed $\mathscr{R}$ in the polynomial ring

$\mathscr{R}[z]$, where z is an indeterminant, and replace $A^{(1,1)}$ by the matrix $zI + A^{(1,1)}$. Since the determinant of $zI + A^{(1,1)}$ is a monic polynomial of degree n, and hence is neither zero nor a zero divisor, (1) holds again. Substituting $z = 0$ (equivalently, equating constant terms of both sides) yields the desired result. ∎

We sketch an alternative proof of Theorem 1 when $\mathscr{R}$ has no zero divisors, a proof suggested to us by the referee. It is based on ideas of **[3]**; see also **[1]**. If $\mathscr{R}$ is a commutative ring with no zero divisors, then we can embed it in its quotient field and then pass to the algebraic closure F. We now regard the blocks $A^{(i,j)}$ as operators on the vector space F^n, and M as an operator on the tensor product $V = F^n \otimes F^k$. Since the blocks $A^{(i,j)}$ commute pairwise, there exists a basis for F^n with respect to which each $A^{(i,j)}$ is upper triangular; see **[2**, p. 108]. We form the tensor product of such a basis with the standard one for F^k, thereby constructing a new basis for V. The change of basis has the effect on M of simultaneously triangularizing each block. Thus it suffices to assume that each block $A^{(i,j)}$ is upper triangular.

The matrix M is permutation-similar to a $n \times n$ block matrix $\tilde{M} = (\tilde{A}_{p,q})$ such that $\tilde{A}_{p,q} = (A^{(i,j)}_{p,q})$ is an $k \times k$ matrix consisting of the p, q-entries of the $A^{(i,j)}$. Since each $A^{(i,j)}$ is upper triangular, $\tilde{A}_{p,q} = 0$ whenever $p > q$. Hence $|\tilde{M}| = |\tilde{A}_{1,1}| \cdots |\tilde{A}_{n,n}| = \prod_{r=1}^{n} \sum_{\pi \in S_k} (\operatorname{sgn} \pi) A^{(1,\pi(1))}_{r,r} \cdots A^{(k,\pi(k))}_{r,r}$. Since each $A^{(i,j)}$ is upper triangular, the last product is equal to $\prod_{r=1}^{n} \sum_{\pi \in S_k} (\operatorname{sgn} \pi)(A^{(1,\pi(1))} \cdots A^{(k,\pi(k))})_{r,r}$. But this is equal to $|\sum_{\pi \in S_k} (\operatorname{sgn} \pi) A^{(1,\pi(1))} \cdots A^{(k,\pi(k))}|$. Hence (1) holds.

The second proof shows that the commutativity hypotheses of Theorem 1 can be replaced by the weaker condition that the blocks can be simultaneously triangularized. However, *some* hypothesis about the blocks is certainly needed for the conclusion of the theorem to hold, as the reader can see by considering the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We conclude by describing a class of block matrices that satisfy the commutativity hypothesis of Theorem 1. Matrices of this type arose in **[5]**, and were the original motivation for this investigation. Let $p^{(i,j)}(t)$ be polynomials, $1 \leq i,j \leq k$, and let N be an $n \times n$ matrix. All coefficients are in $\mathscr{R}$, which can be taken to be the field of complex numbers, if the reader desires. Since the matrices $p^{(i,j)}(N)$ commute pairwise, the block matrix

$$M = \begin{pmatrix} p^{(1,1)}(N) & \cdots & p^{(1,k)}(N) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(N) & \cdots & p^{(k,k)}(N) \end{pmatrix}$$

satisfies the hypothesis of Theorem 1. In fact, using the theorem we can say something about the determinant of M. Let $p(t)$ be the determinant of

$$\begin{pmatrix} p^{(1,1)}(t) & \cdots & p^{(1,k)}(t) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(t) & \cdots & p^{(k,k)}(t) \end{pmatrix},$$

and let $\zeta_1, \ldots, \zeta_n$ be the (not necessarily distinct) eigenvalues of N. We leave the proof of the following assertion as an exercise:

$$|M| = \prod_{r=1}^{n} p(\zeta_r).$$

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Mixtilinear Incircles

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L. Bankoff [**1**] has coined the term *mixtilinear incircles* of a triangle for the three circles each tangent to two sides and to the circumcircle internally. Consider a triangle ABC and its mixtilinear incircle in the angle A, with center K_A, and radius ρ_A. Bankoff has established the fundamental formula

$$r = \rho_A \cdot \cos^2 \frac{\alpha}{2}, \tag{1}$$

where r is the inradius of the triangle, and α is the magnitude of the angle at A. This formula had appeared earlier as an exercise in [**2**, p. 23]. It leads to a simple construction of the mixtilinear incircle. Denote by I the incenter of triangle ABC, and let the perpendicular through I to the bisector of angle A intersect the sides AC, AB at Y_1 and Z_1, respectively. The perpendiculars at these points to their respective sides intersect again on the angle bisector, at the mixtilinear incenter K_A. The circle with center K_A, passing through Y_1 (and Z_1), is the mixtilinear incircle in angle A; see Figure 1.

In this note, we demonstrate the usefulness of the notion of barycentric coordinates in discovering remarkable geometric properties relating to the mixtilinear incircles of a triangle. To keep the note self-contained, we refrain from using (1), except for the remarks at the end.

Denote by A' the point of contact of the mixtilinear incircle in angle A with the circumcircle. For convenience, we denote K_A by K, and ρ_A by ρ when there is no danger of confusion; see Figure 2. The center K lies on the bisector of angle A, and $AK : KI = \rho : -(\rho - r)$. In terms of barycentric coordinates,

$$K = \frac{1}{r}[-(\rho - r)A + \rho I]. \tag{2}$$