10716. *Proposed by Michael L. Catalano-Johnson and Daniel Loeb, Daniel Wagner Associates, Malvern, PA.* What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

10717. *Proposed by Marcin Mazur, University of Chicago, Chicago, IL.* We say that a tetrahedron is *rigid* if it is determined by its volume, the areas of its faces, and the radius of its circumscribed sphere. We say that a tetrahedron is *very rigid* if it is determined just by the areas of its faces and the radius of its circumscribed sphere.
(**a**) Prove that every tetrahedron with faces of equal area is rigid.
(**b**) Prove that a very rigid tetrahedron with faces of equal area is regular.
(**c**)* Is every tetrahedron rigid?
(**d**)* Is every very rigid tetrahedron regular?

SOLUTIONS

Subtracting Square Roots Repeatedly

10568 [1997, 68]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let n be a nonnegative integer. The sequence defined by $x_0 = n$ and $x_{k+1} = x_k - \lceil\sqrt{x_k}\rceil$ for $k \ge 0$ converges to 0. Let $f(n)$ be the number of steps required; i.e., $x_{f(n)} = 0$ but $x_{f(n)-1} > 0$. Find a closed form for $f(n)$.

Solution by Denis Constales, University of Gent, Gent, Belgium. Every positive integer n can be written uniquely in the form $p^2 - q$, where p and q are integers satisfying $p \ge 1$ and $0 \le q \le 2p - 2$ (take $p = \lceil\sqrt{n}\,\rceil$ and $q = p^2 - n$). We call this *standard form* for n. We obtain the desired formula in terms of these parameters p and q.

Using standard form, let $n' = n - \lceil\sqrt{n}\,\rceil = p^2 - (q + p)$. We distinguish two cases. **Case 1:** $p - 1 \le q \le 2p - 2$. We rewrite n' as $(p-1)^2 - (q - (p-1))$. Since $q \ge p - 1$, this expresses n' in standard form with $p' = p - 1$ and $q' = q - (p-1)$ (when $p > 2$). **Case 2:** $0 \le q \le p - 1$. Now $n' = p^2 - (q + p)$ is standard form for n' with $p' = p$ and $q' = q + p$. The next value $n'' = n' - \lceil\sqrt{n'}\,\rceil = p^2 - (q + 2p)$. Expressed in standard form, this is $n'' = (p-1)^2 - (q+1)$ (when $p > 2$).

We have applied the transformation once in Case 1 and twice in Case 2. Thus

$$f(p^2 - q) = \begin{cases} 2 + f((p-1)^2 - (q+1)) & \text{if } 0 \le q \le p - 2 \\ 1 + f((p-1)^2 - (q - p + 1)) & \text{if } p - 1 \le q \le 2p - 2 \end{cases}$$

whenever $p > 2$ and $0 \le q \le 2p - 2$. The cases $p \le 2$ occur for $n \in \{1, 2, 3, 4\}$, where $f(n) = 1, 1, 2, 2$, respectively. With the recurrence, these initial conditions define f. Our closed form is

$$f(p^2 - q) = \begin{cases} 2p - \lfloor\log_2(p+q)\rfloor - 1 & \text{if } 0 \le q \le p - 1 \\ 2p - \lfloor\log_2 q\rfloor - 2 & \text{if } p \le q \le 2p - 2 \end{cases}$$

for integers p, q such that $1 \le p$ and $0 \le q \le 2p - 2$. Also, we set $f(0) = 0$.

The proof of the formula is immediate by induction, using the recurrence in the three cases $0 \le q \le p - 2$, $q = p - 1$, and $p \le q \le 2p - 2$. The only simplification needed occurs in the second case, where $\lceil\log_2(2p-1)\rceil = 1 + \lceil\log_2(p-1)\rceil$, which follows immediately when $p > 1$.

Editorial comment. Robin J. Chapman and the GCHQ Problems Group expressed $f(n)$ using the single formula $f(n) = \lfloor 4n + 2^{m+3} - 3\rfloor - (m+2)$, where $m = \lfloor\log_2(\sqrt{n} + 1)\rfloor$.

Solved also by T. Amdeberhan, K. L. Bernstein, R. J. Chapman (U. K.), D. A. Darling, M. N. Deshpande & N. N. Kasturiwale (India), K. Ferguson, R. Holzsager, W. Janous (Austria), F. Kemp, P. G. Kirmser, N. Komanda, Y. Kong, J. H. Lindsey II, W. A. Newcomb, C. R. Pranesachar (India), K. Schilling, J. H. Steelman, D. Trautman, X. Wang, D. Yuen, GCHQ Problems Group (U. K.), Westmont Problems Group, and the proposer.