Graphs without Increasing Paths

10572 [1997, 168]. *Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA.* Let $f(n)$ be the number of graphs (without loops or multiple edges) on the vertices $1, 2, \ldots, n$ such that no path of length two has vertices i, j, k (in that order) with $i < j < k$. Let $g(n)$ be the total number of subspaces of an n-dimensional vector space over a 2-element field. show that

$$\sum_{n \geq 0} f(n)\frac{x^n}{n!} = e^{-x} \sum_{n \geq 0} g(n)\frac{x^n}{n!}.$$

Solution by Stephen C. Locke, Florida Atlantic University, Boca Raton, FL. Let V be an n-dimensional binary vector space, and let S be a subspace of V. We may take S to be the row space of an $m \times n$ binary matrix M. Furthermore, we may assume that M is row-reduced, so that the leading 1 in each row is the only 1 in its column. We call these entries *pivots*.

Construct a graph G whose vertices are the nonzero columns of M (G is empty when S has dimension 0). For each pivot element $m_{i,j}$, the vertex representing column j is adjacent to the vertex representing column r if $m_{i,r} \neq 0$. Thus all edges consist of a pivot column and a higher-indexed non-pivot column. In particular, G has no path i, j, k with $i < j < k$. Furthermore, the row-reduced matrix M and thus S can be retrieved from G.

If we relabel the vertices of G with $1, 2, \ldots, k$ preserving the order of the original labels, then the new graph is of the type counted by $f(k)$. Thus there are $\sum_{k=0}^{n} \binom{n}{k} f(k)$ such graphs, and the bijection with subspaces yields $g(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)$.

When multiplying power series, the coefficient of $x^n/n!$ in the product of $\sum_{n \geq 0} a_n x^n/n!$ and $\sum_{n \geq 0} b_n x^n/n!$ is $\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Thus $\sum_{n \geq 0} g(n)x^n/n! = e^x \sum_{n \geq 0} f(n)x^n/n!$.

Solved also by D. Beckwith, D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), R. Ehrenborg, R. Holzsager, D. E. Knuth, L. Pebody (U. K.), and the proposer.

A Card-Matching Game

10576 [1997, 169]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Alice and Bill have identical decks of 52 cards. Alice shuffles her deck and deals the cards face up into 26 piles of two cards each. Bill does the same with his deck. If any one of Alice's top cards exactly matches any of Bill's, the matching cards are removed. Play continues until none of the cards on top of Alice's piles matches any of the cards on top of Bill's piles. What is the probability that all 52 pairs of cards will be matched?

Solution by Philip D. Straffin, Beloit College, Beloit, WI. Given a particular deal of $2n$ cards for Alice, let T_n be the number of possible deals for Bill, let W_n be the number of these that succeed (all cards are matched), and let L_n be the number that lose. We compute W_n/T_n.

Given a deal for Alice, a game is specified by Bill's piles: a partition of $\{1, 2, \ldots, 2n\}$ into pairs and a choice for the top card in each pair. After specifying which of Bill's cards is paired with 1 and which is the top card in this pair, there are T_{n-1} ways to complete the deal. Hence $T_n = 2(2n-1)T_{n-1}$.

Now consider L_n. As long as either player has any single-card piles, the game is not lost, since more than half of that player's cards are exposed and there must be a match. Hence when a game is lost, each player retains $k \geq 1$ piles of two cards, and Bill's top cards must be exactly Alice's bottom cards. The piles that were removed were a successful game of size $n - k$. Since the hidden cards in Bill's blocked piles are chosen from Alice's top cards and can be arranged in $k!$ ways, we have $L_n = \sum_{k=1}^{n} \binom{n}{k} k!\, W_{n-k}$.

Since $T_n = W_n + L_n$, we have $T_n = \sum_{k=0}^{n} \binom{n}{k} k!\, W_{n-k}$. Thus

$$nT_{n-1} = n\sum_{k=0}^{n-1} \binom{n-1}{k} k!\, W_{n-1-k} = \sum_{k=0}^{n-1} \binom{n}{k+1}(k+1)!\, W_{n-1-k} = \sum_{k=1}^{n} \binom{n}{k} k!\, W_{n-k} = L_n.$$

With the recurrence for T_n, this yields

$$\frac{L_n}{T_n} = \frac{nT_{n-1}}{2(2n-1)T_{n-1}} = \frac{n}{4n-2},$$

and the probability of matching all cards is $(3n-2)/(4n-2)$. When $n = 26$, this is about 0.745.

Editorial comment. The GCHQ Problems Group considered a generalization: Alice starts with a piles of two cards (doubles) and $c - 2a$ piles of one card (singles), while Bill starts with b doubles and $c-2b$ singles. The probability of failure is $ab/(c(c-1))$, for $c \geq 2$. The proposer observed that his problem is a variation on one that Lewis Carroll recorded in his diary on February 29, 1856. That problem, called Sympathy, was given to him by someone named Pember and remains unsolved. In Sympathy, the cards are dealt into 18 piles of sizes $3, 3, \ldots, 3, 1$ instead of 26 piles of size 2.

Solved also by D. Beckwith, D. Callan, R. J. Chapman (U. K.), D. A. Darling, J. E. Dawson (Australia), R. Ehrenborg, P. Griffin, C. M. Grinstead, V. Hernández & J. Martín (Spain), R. Holzsager, M. A. Javaloyes Victoria (Spain), J. T. Lee, J. H. Lindsey II, J. H. Nieto (Venezuela), L. Pebody (U. K.), B. Peterson, M. A. Prasad (India), A. L. Rocha, W. J. Seaman, D. S. Silver & S. G. Williams, M. Woltermann, N. Zoroa & P. Zoroa (Spain), GCHQ Problems Group (U. K.), and the proposer.

Wilson's Theorem in Disguise

10578 [1997, 270]. *Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.* Consider the sequence $y_2, y_3, \ldots$ defined by the recurrence relation

$$(n+1)(n-2)y_{n+1} = n(n^2-n-1)y_n - (n-1)^3 y_{n-1}$$

and initial conditions $y_2 = y_3 = 1$. Show that y_n is an integer if and only if n is prime.

Solution by Florian Herzig, Perchtoldsdorf, Austria. Let $x_n = ny_n$ for all $n \geq 2$. We have $x_2 = 2$, $x_3 = 3$, and $(n-2)x_{n+1} = (n^2-n-1)x_n - (n-1)^2 x_{n-1}$ for $n \geq 3$, which becomes

$$\frac{x_{n+1}-x_n}{n-1} = (n-1) \cdot \frac{x_n - x_{n-1}}{n-2}.$$

Setting $z_n = (x_{n+1}-x_n)/(n-1)$ yields $z_n = (n-1)!$, and so $x_{n+1} - x_n = (n-1)z_n = (n-1)(n-1)! = n! - (n-1)!$. Hence

$$x_n = x_2 + \sum_{k=2}^{n-1}(x_{k+1}-x_k) = 2 + (n-1)! - 1! = (n-1)! + 1.$$

It follows that $y_n = x_n/n = ((n-1)!+1)/n$. By Wilson's Theorem, $(n-1)!+1$ is divisible by n if and only if n is prime. Hence y_n is an integer if and only if n is prime.

Editorial comment. In some books, Wilson's Theorem is the statement that $(n-1)!+1$ is divisible by n when n is prime. The converse is also well known and is easily established, since $ny_n = (n-1)!+1$ requires that n and $(n-1)!$ be relatively prime.

Walther Janous noted that one might also study sequences of the form $y_n(a) = \frac{n!+a}{n+a}$ for any integer a. He asks whether there are any integers $a > 1$ for which this sequence contains infinitely many integers; either answer suggests other interesting questions.

Solved also by R. Akhlaghi & F. Sami, J. Anglesio (France), M. N. Balachandran (India), R. Barbara (Lebanon), C. Berg (Sweden), E. Brown, M. Burger (Austria), S. Butcher & X. Wang, D. Callan, R. J. Chapman (U. K.), M. P. Chernesky, B. Conolly (U. K.), D. A. Darling, J. E. Dawson (Australia), D. Donini (Italy), H. Gauchman, C. Georghiou (Greece), R. Heller, R. Holzsager, T. Jager, W. Janous (Austria), W. Kim (South Korea), R. A. Kopas, J. H. Lindsey II, S. C. Locke, R. Martin (Germany), V. J. Matsko, B. McCabe, J. H. Nieto (Venezuela), R. Padma (India), M. D. Pearce, W. H. Pierce, J. Robertson, R. K. Schwartz, Z. Shan & E. T. H. Wang (Canada), P. Simeonov, A. Sinefakopoulos (Greece), N. C. Singer, A. Stenger, D. C. Terr, A. Tissier (France), J. Van hamme (Belgium), J. H. van Lint (The Netherlands), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.