**Wilson's Theorem in Disguise: 10578**

Herbert S. Wilf; Florian Herzig

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With the recurrence for $T_n$, this yields

$$\frac{L_n}{T_n} = \frac{nT_{n-1}}{2(2n-1)T_{n-1}} = \frac{n}{4n-2},$$

and the probability of matching all cards is $(3n-2)/(4n-2)$. When $n = 26$, this is about 0.745.

*Editorial comment.* The GCHQ Problems Group considered a generalization: Alice starts with $a$ piles of two cards (doubles) and $c - 2a$ piles of one card (singles), while Bill starts with $b$ doubles and $c-2b$ singles. The probability of failure is $ab/(c(c-1))$, for $c \geq 2$. The proposer observed that his problem is a variation on one that Lewis Carroll recorded in his diary on February 29, 1856. That problem, called Sympathy, was given to him by someone named Pember and remains unsolved. In Sympathy, the cards are dealt into 18 piles of sizes $3, 3, \ldots, 3, 1$ instead of 26 piles of size 2.

Solved also by D. Beckwith, D. Callan, R. J. Chapman (U. K.), D. A. Darling, J. E. Dawson (Australia), R. Ehrenborg, P. Griffin, C. M. Grinstead, V. Hernández & J. Martín (Spain), R. Holzsager, M. A. Javaloyes Victoria (Spain), J. T. Lee, J. H. Lindsey II, J. H. Nieto (Venezuela), L. Pebody (U. K.), B. Peterson, M. A. Prasad (India), A. L. Rocha, W. J. Seaman, D. S. Silver & S. G. Williams, M. Woltermann, N. Zoroa & P. Zoroa (Spain), GCHQ Problems Group (U. K.), and the proposer.

### Wilson's Theorem in Disguise

**10578** [1997, 270]. *Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.* Consider the sequence $y_2, y_3, \ldots$ defined by the recurrence relation

$$(n+1)(n-2)y_{n+1} = n(n^2-n-1)y_n - (n-1)^3 y_{n-1}$$

and initial conditions $y_2 = y_3 = 1$. Show that $y_n$ is an integer if and only if $n$ is prime.

*Solution by Florian Herzig, Perchtoldsdorf, Austria.* Let $x_n = ny_n$ for all $n \geq 2$. We have $x_2 = 2$, $x_3 = 3$, and $(n-2)x_{n+1} = (n^2-n-1)x_n - (n-1)^2 x_{n-1}$ for $n \geq 3$, which becomes

$$\frac{x_{n+1}-x_n}{n-1} = (n-1)\cdot\frac{x_n - x_{n-1}}{n-2}.$$

Setting $z_n = (x_{n+1}-x_n)/(n-1)$ yields $z_n = (n-1)!$, and so $x_{n+1} - x_n = (n-1)z_n = (n-1)(n-1)! = n! - (n-1)!$. Hence

$$x_n = x_2 + \sum_{k=2}^{n-1}(x_{k+1}-x_k) = 2 + (n-1)! - 1! = (n-1)! + 1.$$

It follows that $y_n = x_n/n = ((n-1)!+1)/n$. By Wilson's Theorem, $(n-1)!+1$ is divisible by $n$ if and only if $n$ is prime. Hence $y_n$ is an integer if and only if $n$ is prime.

*Editorial comment.* In some books, Wilson's Theorem is the statement that $(n-1)!+1$ is divisible by $n$ when $n$ is prime. The converse is also well known and is easily established, since $ny_n = (n-1)!+1$ requires that $n$ and $(n-1)!$ be relatively prime.

Walther Janous noted that one might also study sequences of the form $y_n(a) = \frac{n!+a}{n+a}$ for any integer $a$. He asks whether there are any integers $a > 1$ for which this sequence contains infinitely many integers; either answer suggests other interesting questions.

Solved also by R. Akhlaghi & F. Sami, J. Anglesio (France), M. N. Balachandran (India), R. Barbara (Lebanon), C. Berg (Sweden), E. Brown, M. Burger (Austria), S. Butcher & X. Wang, D. Callan, R. J. Chapman (U. K.), M. P. Chernesky, B. Conolly (U. K.), D. A. Darling, J. E. Dawson (Australia), D. Donini (Italy), H. Gauchman, C. Georghiou (Greece), R. Heller, R. Holzsager, T. Jager, W. Janous (Austria), W. Kim (South Korea), R. A. Kopas, J. H. Lindsey II, S. C. Locke, R. Martin (Germany), V. J. Matsko, B. McCabe, J. H. Nieto (Venezuela), R. Padma (India), M. D. Pearce, W. H. Pierce, J. Robertson, R. K. Schwartz, Z. Shan & E. T. H. Wang (Canada), P. Simeonov, A. Sinefakopoulos (Greece), N. C. Singer, A. Stenger, D. C. Terr, A. Tissier (France), J. Van hamme (Belgium), J. H. van Lint (The Netherlands), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.