Möbius and Riemann

10582 [1997, 270]. *Proposed by Peter Lindqvist and Kristian Seip, Norwegian University of Science and Technology, Trondheim, Norway.* Let $\mu(n)$ denote the Möbius function of number theory, and let $\zeta(s)$ denote the Riemann zeta function. Prove that

$$\zeta(s)\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{(\gcd(m,n))^s}{(mn)^s}\mu(m)\mu(n)=1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2$$

when $s>1$.

Solution by David M. Bradley, University of Maine, Orono, ME. We use the well-known fact that $\sum_{n|j}\mu(n)=0$ for $j\geq 2$. We compute

$$\begin{aligned}1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2&=1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{n|j}\mu(n)-\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right)^2\\&=1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right)^2=1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{m|j\\ m\leq N}}\mu(m)\right)\left(\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right).\end{aligned}$$

In the inner sums, m and n both divide j if and only if $\operatorname{lcm}(m,n)|j$. Writing $j=k\cdot\operatorname{lcm}(m,n)$ and interchanging the order of summation yields

$$1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2=\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\mu(m)\mu(n)}{(\operatorname{lcm}(m,n))^s}\sum_{k=1}^{\infty}\frac{1}{k^s}.$$

Since $\operatorname{lcm}(m,n)=mn/\gcd(m,n)$, the result follows.

Editorial comment. The proposers' solution was quite different. They introduced the functions $f(x)=\sum_{k=1}^{\infty}\sin(kx)/k^s$ and $f_N(x)=\sum_{n=1}^{N}(\mu(n)/n^s)f(nx)$. For $N\in\mathbb{N}$, we have $f_N(x)\to\sin x$ as $s\to\infty$, so it is natural to compute the $L_2(0,\pi)$ norm of the "error" $\sin x-f_N(x)$. Doing this in two different ways yields the result.

Solved also by M. N. Balachandran (India), D. Callan, R. J. Chapman (U. K.), R. Holzsager, J. H. Lindsey II, R. Padma (India), P. Simeonov, NSA Problems Group, and the proposers.

Catalan and Hankel

10585 [1997, 361]. *Proposed by Alta Kellogg, Ormond Beach, FL.* A sequence $a_0, a_1, \ldots$ of real numbers is called *strictly totally positive* (STP) if every submatrix of the Hankel matrix $(a_{i+j})_{i,j\geq 0}$ has positive determinant.

(a) Show that the sequence $C_0, C_1, \ldots$ of Catalan numbers, defined by $C_n=\frac{1}{n+1}\binom{2n}{n}$, is STP.

(b) Show that the sequence of Catalan numbers is minimal in the following sense: If $a_0, a_1, a_2, \ldots$ is an STP sequence of positive integers with $a_n\leq C_n$ for every n, then $a_n=C_n$ for every n.

Solution to part **(a)** *by David Callan, Madison, WI.* Let C be the matrix (C_{i+j}). For sets of indices $\mathbf{u}=\{u_1<\cdots<u_n\}$ and $\mathbf{v}=\{v_1<\cdots<v_n\}$, let $C[\mathbf{u}|\mathbf{v}]$ denote the submatrix of C with rows indexed by $\mathbf{u}$ and columns indexed by $\mathbf{v}$. Recall that the Catalan number C_k is the number of Dyck paths ("mountain ranges") of length $2k$. (A Dyck path consists of northeast and southeast steps, starts on the x axis, ends on the x axis, and never falls below