



Monomial Bounds for Polynomials: 10613

F. J. Flanigan; Roberto Tauraso

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where $\epsilon = (1 + (-1)^s)/2$. We now use the addition formulæ for the cosine to express a product of two sines as a difference of two cosines and simplify to obtain

$$f_{2s}(x) = 2^\epsilon (\sinh(\pi x))^{(-1)^s} \sin(\pi x) \prod_{j=1}^{s-1} \left(\cosh\left(2\pi x \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi x \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j}$$

and hence

$$\begin{aligned} P_{2s}(m) &= \lim_{x \rightarrow m} \frac{m^{2s} + x^{2s}}{m^{2s} - x^{2s}} f_{2s}(x) = 2^\epsilon (\sinh(\pi m))^{(-1)^s} \frac{2\pi m^{2s} \cos(\pi m)}{-2sm^{2s-1}} \times \\ &\quad \prod_{j=1}^{s-1} \left(\cosh\left(2\pi m \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi m \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j} \\ &= (-1)^{m+1} \frac{2^\epsilon \pi m}{s} (\sinh(\pi m))^{(-1)^s} \prod_{j=1}^{s-1} \left(\cosh\left(2\pi m \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi m \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j} \end{aligned}$$

as required. Note that this formula gives $P_2(m) = (-1)^{m+1} \pi m / \sinh(\pi m)$ when $s = 1$.

Solved also by R. J. Chapman (U. K.), K.-K. Choi, R. Mortini (France), H.-J. Seiffert (Germany), and the proposers.

Monomial Bounds for Polynomials

10613 [1997, 767]. *Proposed by F. J. Flanigan, San Jose State University, San Jose, CA.* Fix a positive real number ν . Find all polynomials $P(x)$ with nonnegative real coefficients such that

- (a) $P(0) = 0$, $P(1) = 1$, and $P(x) \leq x^\nu$ for all $x \geq 0$.
 (b) $P(0) = 0$, $P(1) = 1$, and $P(x) \geq x^\nu$ for all $x \geq 0$.

Solution by Roberto Tauraso, Firenze, Italy. Let $P(x) = \sum_{i=m}^n a_i x^i$ with nonnegative real coefficients, $a_m > 0$, and $a_n > 0$. The conditions $P(0) = 0$, $P(1) = 1$ imply immediately that $m \geq 1$ and $\sum_{i=m}^n a_i = 1$.

(a) If $P(x) \leq x^\nu$ for all $x \geq 0$, then necessarily

$$\lim_{x \rightarrow +\infty} \frac{P(x)}{x^\nu} = \lim_{x \rightarrow +\infty} \frac{a_n x^n}{x^\nu} \leq 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{P(x)}{x^\nu} = \lim_{x \rightarrow 0^+} \frac{a_n x^m}{x^\nu} \leq 1,$$

which imply $n \leq \nu$ and $\nu \leq m$, respectively. Hence $m = \nu = n$, and condition (a) is satisfied if and only if ν is a positive integer and $P(x) = x^\nu$.

(b) If $P(x) \geq x^\nu$ for all $x \geq 0$, then the function $\varphi(x) = P(x) - x^\nu$ is nonnegative, differentiable, and satisfies $\varphi(1) = 0$. Hence φ has a minimum at $x = 1$, so $\varphi'(1) = (\sum_{i=m}^n i a_i) - \nu = 0$. Thus ν is a convex combination of the integers m, \dots, n .

On the other hand, suppose that a polynomial $P(x) = \sum_{i=m}^n a_i x^i$ has nonnegative real coefficients such that $\sum_{i=m}^n a_i = 1$ and $\sum_{i=m}^n i a_i = \nu$. Then $P(0) = 0$, $P(1) = 1$, and, by the weighted arithmetic-geometric mean inequality, $P(x) = \sum_{i=m}^n a_i x^i \geq x^\nu$ for all $x \geq 0$. Thus condition (b) is satisfied if and only if $\nu \geq 1$ and $P(x) = \sum_{i=m}^n a_i x^i$, with $\sum_{i=m}^n a_i = 1$ and $\sum_{i=m}^n i a_i = \nu$.

Editorial comment. Erik I. Verriest provided a generalization to the case in which $P(x)$ is a power series. The results are the same as in the selected solution, except that in part (b) the upper limit of summation n may be infinite.

Solved also by P. Alsholm (Denmark), K. F. Andersen (Canada), T. Armstrong, M. Babilonová & J. Kupka (Czech Republik), R. J. Chapman (U. K.), J. H. Lindsey II, A. Nijenhuis, C. Popescu (Belgium), H.-J. Seiffert (Germany), E. I. Verriest, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.