where $\epsilon = (1+(-1)^s)/2$. We now use the addition formulæ for the cosine to express a product of two sines as a difference of two cosines and simplify to obtain

$$f_{2s}(x) = 2^{\epsilon}(\sinh(\pi x))^{(-1)^s}\sin(\pi x)\prod_{j=1}^{s-1}\left(\cosh\big(2\pi x\sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi x\cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$

and hence

$$P_{2s}(m) = \lim_{x\to m}\frac{m^{2s}+x^{2s}}{m^{2s}-x^{2s}}f_{2s}(x) = 2^{\epsilon}\big(\sinh(\pi m)\big)^{(-1)^s}\frac{2\pi m^{2s}\cos(\pi m)}{-2sm^{2s-1}}\times$$
$$\prod_{j=1}^{s-1}\left(\cosh\big(2\pi m\sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi m\cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$
$$=(-1)^{m+1}\frac{2^{\epsilon}\pi m}{s}\big(\sinh(\pi m)\big)^{(-1)^s}\prod_{j=1}^{s-1}\left(\cosh\big(2\pi m\sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi m\cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$

as required. Note that this formula gives $P_2(m) = (-1)^{m+1}\pi m/\sinh(\pi m)$ when $s = 1$.

Solved also by R. J. Chapman (U. K.), K.-K. Choi, R. Mortini (France), H.-J. Seiffert (Germany), and the proposers.

Monomial Bounds for Polynomials

10613 [1997, 767]. *Proposed by F. J. Flanigan, San Jose State University, San Jose, CA.* Fix a positive real number ν. Find all polynomials $P(x)$ with nonnegative real coefficients such that

(**a**) $P(0) = 0$, $P(1) = 1$, and $P(x) \le x^{\nu}$ for all $x \ge 0$.
(**b**) $P(0) = 0$, $P(1) = 1$, and $P(x) \ge x^{\nu}$ for all $x \ge 0$.

Solution by Roberto Tauraso, Firenze, Italy. Let $P(x) = \sum_{i=m}^{n} a_i x^i$ with nonnegative real coefficients, $a_m > 0$, and $a_n > 0$. The conditions $P(0) = 0$, $P(1) = 1$ imply immediately that $m \ge 1$ and $\sum_{i=m}^{n} a_i = 1$.
(**a**) If $P(x) \le x^{\nu}$ for all $x \ge 0$, then necessarily

$$\lim_{x\to+\infty}\frac{P(x)}{x^{\nu}} = \lim_{x\to+\infty}\frac{a_n x^n}{x^{\nu}} \le 1 \qquad \text{and} \qquad \lim_{x\to 0^+}\frac{P(x)}{x^{\nu}} = \lim_{x\to 0^+}\frac{a_n x^m}{x^{\nu}} \le 1,$$

which imply $n \le \nu$ and $\nu \le m$, respectively. Hence $m = \nu = n$, and condition (**a**) is satisfied if and only if ν is a positive integer and $P(x) = x^{\nu}$.
(**b**) If $P(x) \ge x^{\nu}$ for all $x \ge 0$, then the function $\varphi(x) = P(x) - x^{\nu}$ is nonnegative, differentiable, and satisfies $\varphi(1) = 0$. Hence φ has a minimum at $x = 1$, so $\varphi'(1) = \left(\sum_{i=m}^{n} i a_i\right) - \nu = 0$. Thus ν is a convex combination of the integers $m, \dots, n$.

On the other hand, suppose that a polynomial $P(x) = \sum_{i=m}^{n} a_i x^i$ has nonnegative real coefficients such that $\sum_{i=m}^{n} a_i = 1$ and $\sum_{i=m}^{n} i a_i = \nu$. Then $P(0) = 0$, $P(1) = 1$, and, by the weighted arithmetic-geometric mean inequality, $P(x) = \sum_{i=m}^{n} a_i x^i \ge x^{\nu}$ for all $x \ge 0$. Thus condition (**b**) is satisfied if and only if $\nu \ge 1$ and $P(x) = \sum_{i=m}^{n} a_i x^i$, with $\sum_{i=m}^{n} a_i = 1$ and $\sum_{i=m}^{n} i a_i = \nu$.

Editorial comment. Erik I. Verriest provided a generalization to the case in which $P(x)$ is a power series. The results are the same as in the selected solution, except that in part (**b**) the upper limit of summation n may be infinite.

Solved also by P. Alsholm (Denmark), K. F. Andersen (Canada), T. Armstrong, M. Babilonová & J. Kupka (Czech Republik), R. J. Chapman (U. K.), J. H. Lindsey II, A. Nijenhuis, C. Popescu (Belgium), H.-J. Seiffert (Germany), E. I. Verriest, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.