The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians

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1. INTRODUCTION. If you pick up nearly any elementary ordinary differential equations text or calculus text, you are likely to find a short section, or at least a problem, on the motion of a body subject to some type of drag force along with a calculation of the body's terminal velocity. Two favorite examples seem to be the motion of a projectile like a baseball and the motion of a skydiver/parachutist, both through the air. By a *skydiver* we mean a person falling without his parachute open. Most textbook authors model the motion of these objects using a drag force that depends linearly in the velocity. Unfortunately, the physical assumption about the linear dependence of the drag force on velocity is often incorrect, and thus the model's predictions are physically implausible.

In particular it was surprising to see the faulty linear resistance model for a parachutist's velocity used in the popular calculus reform text by Hughes-Hallet, Gleason, et al. [**9**, p. 515], since the reform movement prides itself on concern for realistic applications. The first edition of this text even supplied non-referenced *observed data* to fit its linear model. The authors state "The fact that there is good agreement between the observed and predicted data suggest that our assumption about the air resistance is reasonable." The recent second edition omits the table of *observed data* but not the flawed model.

The purpose of this note is to explain the dependence of the drag force on velocity for a general mathematical audience and to present a few realistic models. Section 5 contains an interesting model (with a closed form solution) for re-entry of the space shuttle into the earth's atmosphere.

Dimensional analysis is an important tool in aerodynamics and fluid dynamics, and can be used to obtain key results (5) and (6). To help give mathematicians some insight into the spirit of this important technique, we present in Section 6 an amusing application of dimensional analysis to prove the Pythagorean Theorem.

The science of modeling drag is more physical and empirical than mathematical, and it relies on the results of many wind-tunnel experiments. There has been a significant amount of *theoretical* work in the engineering literature, but few of the results can be considered completely rigorous by mathematical standards. There are also large gaps in our understanding of basic properties of the Navier Stokes equation. In particular, there are important unsolved problems on the large-time existence and uniqueness of solutions of the Navier-Stokes equation in three dimensions.

For detailed information on the aerodynamics and fluid mechanics pertinent to this paper, see [**7**], [**8**], [**11**], [**12**], [**19**], and [**22**].

2. THE BASIC EQUATIONS OF MOTION. Any body moving through a fluid such as water or air creates a drag force that tends to retard its motion. Such motion is usually described by the Navier-Stokes (nonlinear partial differential) equations. In elementary textbooks, the motion is always assumed to be one dimensional, e.g., the ball is dropped and the skydiver has no horizontal movement and there is no wind. We observe in Section 4 that this assumption does not permit modeling of a modern parachute. Many (if not most) elementary mathematics textbook authors assume that the drag force for a baseball or skydiver/parachutist moving in air is proportional to the velocity v of the falling body, and at least one leading freshman physics text makes this assumption. This leads to the linear differential equation of motion

$$m\frac{dv}{dt} = mg - k_1 v, \tag{1}$$

where k_1 is a constant (whose physical meaning is rarely discussed), m is the mass of the body, and g is the gravitational constant. This linear differential equation can be solved easily to obtain the body's velocity as a function of time, beginning at rest:

$$v_1(t) = \frac{mg}{k_1}(1 - \exp(-k_1\, t/m)), \quad v_1(0) = 0. \tag{2}$$

The terminal velocity is $\lim_{t\to\infty} v_1(t) = mg/k_1$. This terminal velocity is just the equilibrium solution of (1) and could have been obtained easily directly from (1) without explicitly solving the differential equation since physically, the terminal velocity corresponds to the motion when the drag force *precisely equals* the weight mg of the falling object. Any such simple model is necessarily a great simplification of the Navier-Stokes equations for the actual motion of a ball or skydiver. While this model may be correct for bodies that are falling in a vat of heavy oil or for tiny particles of dust or aerosol in air, it is *grossly incorrect* for large bodies falling in air.

Calculations predict and experiments confirm that in air, the drag force on a ball or a skydiver/parachutist can be well *approximated* by a force that is proportional to the *square* of the velocity v^2 (and *not* to the velocity v). The v^2 model for the drag force leads to the nonlinear equation of motion

$$m\frac{dv}{dt} = mg - k_2 v^2, \tag{3}$$

where k_2 is a constant. This is a separable equation, which can be solved easily to obtain the body's velocity as a function of time, beginning at rest:

$$v_2(t) = \sqrt{\frac{mg}{k_2}}\tanh\left(t\sqrt{\frac{k_2 g}{m}}\right), \quad v_2(0) = 0. \tag{4}$$

The terminal velocity is $\lim_{t\to\infty} v_2(t) = \sqrt{mg/k_2}$, which is just the equilibrium solution of (3) and could have been obtained easily directly from (3).

Table 1 contains the experimentally determined terminal velocities for various objects moving through the air. There is a wide range of values for the terminal velocity of a skydiver because the terminal velocity strongly depends on his body position and is considerably higher (almost by a factor of two) during a head first *nose dive* or *feet first* dive than during a fall in the *spread eagle belly-to-Earth* position. The former positions are highly unstable and are difficult to maintain for more than a few seconds. In order to minimize the strong shock to the body at deployment, beginners typically reduce their free fall speed to about 50 m/s before deploying their parachute.

TABLE 1. Approximate terminal velocities for various objects (from Table 9.1 in [4])

Object	Weight (kg)	Terminal Velocity (m/s)
iron ball (shot)	7.3	145
Skydiver	72.6 + 19 (equipment) = 91.6	45 to 80 +
Football	0.41	45
Baseball	0.15	42
Golfball	0.05	40
Softball	0.18	80
Tennis ball	0.06	36
Basketball	0.6	20
Ping-Pong ball	0.003	9
Parachutist (round canopy)	72.6 + 19 (equipment) = 91.6	5

3. SMALL AND LARGE REYNOLDS NUMBERS FLOWS. In general, the drag force depends on many factors including the density and viscosity of the fluid, and the geometry, surface material, surface regularity, and velocity of the body. The dimensionless *Reynolds number* of the fluid plays a key role in determining the drag force, and is defined by

$$R = \frac{\rho d v}{\mu}, \quad \text{or} \quad R = \frac{dv}{\nu},$$

where ρ is the density of the fluid, v is the velocity of the body in the fluid, μ is the viscosity of the fluid, $\nu = \mu/\rho$, and d is a characteristic length (see Table 2). This characteristic length could be a radius, a diameter, a chord length, a body length, etc. depending on what aspect of the problem one is studying. Note that a slowly moving object may have a large Reynolds number if the object is large or the viscosity is small. Turbulent flows are typically associated with large Reynolds numbers, while laminar flows are typically associated with small Reynolds numbers.

TABLE 2. Typical Reynolds numbers for various objects moving in air

Object	Characteristic Length	Typical Reynolds Number
Submarine	Length	300,000,000
Small aircraft	Chord	5,000,000
Parachutist	Diameter	2,500,000
Skydiver	Diameter	1,000,000
Baseball	Diameter	250,000
Model airplane	Chord	50,000
Butterfly	Chord	7,000
Dust particle	Diameter	1

If the Reynolds number is *small*, meaning $R \ll 1$, the Navier-Stokes equation is considerably simplified and the equation of motion reduces to a linear partial differential equation. Strictly speaking, one should assume $R \ll 1$, but the approximation is often reasonable for $R \approx 1$. Stokes analyzed this linear differential equation and found the following formula for the drag force, F_D, on a sphere of radius r moving in the fluid [**21**, p. 217]:

$$F_D = 6\pi\mu r v. \tag{5}$$

This expression is exact in the limit as the Reynolds number goes to zero. Thus, the drag force is proportional to the velocity and the radius of the sphere. Since the fluid density does not appear in the linear partial differential equation, the form of formula (5) can also be obtained with simple dimensional arguments: if the drag force depends only on μ, r, and v, one shows the only function of these quantities that has the units of force is $F_D = C\mu rv$, where C is a constant. A rigorous argument can be based on the Pi theorem of Vaschy and Buckingham [**1**, p. 42], [**2**], [**3**].

Formula (5) can be extended to flows with non-zero Reynolds number. Using techniques in the theory of matched asymptotic expansions, the Stokes approximation (5) can be improved [**17**] to an asymptotic expansion of the form

$$F_D = 6\pi\mu rv\left(1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + O(R^2)\right).$$

Table 3 contains the values of μ and ν for oil, water, and dry air at 100° F. It is known that the viscosity of oils increases rapidly with decreasing temperature.

TABLE 3. Typical values of μ and ν at 100 degrees F

Medium	μ (kg/m sec)	ν (m^2/ sec)
castor oil	0.29	2.8×10^{-4}
water	0.686×10^{-3}	0.691×10^{-6}
dry air	0.19×10^{-4}	1.9×10^{-5}

It has been determined experimentally that (5) is valid for Reynolds numbers $R < 1$ and that a similar dependence occurs in this range of R for bodies with other shapes, i.e., the drag force

$$F_D \approx \text{constant} \times \mu v,$$

where the constant is independent of v. This can be rewritten as $F_D = kv$, where $k = \text{constant} \times \mu$ (see (1) and (2)). From Table 2 we see that baseballs and skydivers/parachutists have $R \gg 1$.

There are some interesting implications of low Reynolds number flows in biology. In particular, [**20**] describes the role of terminal velocity in pollen dispersal, while [**6**] answers the question "Why are there so few aerial plankton?" by explaining how high atmospheric terminal velocities confound the ability of turbulence to keep organisms afloat.

Although there are interesting flows where the drag depends linearly on velocity, they are typically associated with small objects such as raindrops, dust particles, etc. The book [**14**] contains a discussion of modeling falling raindrops over a wide range of Reynolds numbers.

When the Reynolds number is large, but not *too* large, the flow may remain laminar. These cases can be studied using the Navier-Stokes equations in the thin boundary layer around the body where this flow is assumed to be laminar. The resulting equations are called Prandtl's equations [**11**] and one can conclude that (at least for a certain range of R) the drag force is independent of the viscosity. One then uses facts about the Bernoulli equation or dimensional analysis to conclude that

$$F_D \approx \text{constant} \times \rho A v^2, \tag{6}$$

where the constant depends only on the shape and surface characteristics of the body. Numerous experiments in wind tunnels and in aircraft flight tests during the past 80 years have verified that this formula is valid for Reynolds numbers between 3×10^2 and 3×10^5.

For flows in the range of Reynolds numbers, it is customary to introduce the *drag coefficient*, C_D, which is the dimensionless quantity defined by

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho A v^2}. \tag{7}$$

With this definition and (6), we have C_D = constant, i.e., the drag coefficient depends only on the shape and surface characteristics of the body and the Reynolds number. Thus in (3) and (4), the constant $k = \frac{1}{2}C_D \rho A$. Furthermore, the *dynamic pressure* $q \equiv \frac{1}{2}\rho v^2$ plays a fundamental role in aerodynamic theory **[7]**. For instance, when the space shuttle Challenger exploded, it was operating in a high dynamic pressure regime. Very fast aircraft need to operate at high altitude (where ρ is relatively small) to avoid excessive dynamic pressure and catastrophic damage to the aircraft.

For smooth spheres having Reynolds numbers in the range 10^3 to 3×10^5, the drag coefficient is approximately 0.47, while for Reynolds numbers greater than 3×10^5, the drag coefficient is approximately 0.20 (see Figure 1). The text **[22]** contains a good exposition of *sphere drag* for R between 1 and 10^6.

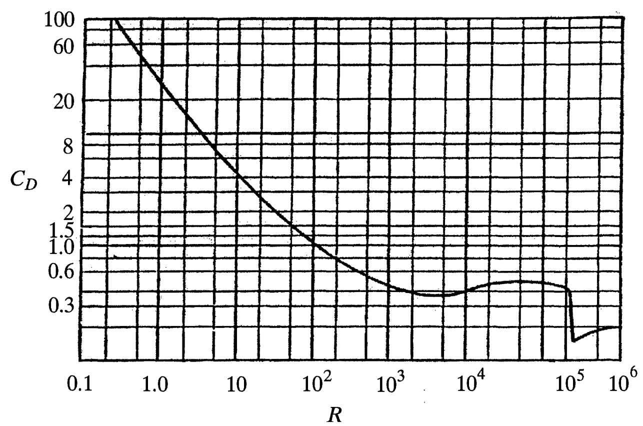

Figure 1. Drag coefficient C_D for a sphere as a function of Reynolds number R (from Figure 34 in **[11]**)

It follows from (4) and (7) that the terminal velocity for a sphere falling in air is approximately

$$v_T = \sqrt{\frac{2W}{\rho C_D \pi r^2}},$$

where W is the weight of the sphere, ρ is the density of air at sea level, and r is the radius of the sphere. The density ρ is a complicated function of temperature, humidity, and pressure (which varies with altitude) so this equation is only an approximation.

The motion of a baseball, with its rough surface, is actually considerably more complicated to model accurately than the motion of a smooth sphere **[13]**.

4. SKYDIVING AND PARACHUTING. We now discuss the motion of a skydiver and a parachutist; useful technical references are **[8]**, **[10]**, **[15]**, and **[16]**. Just as for a sphere falling in air, the terminal velocity for a skydiver is approximately

$$v_T = \sqrt{\frac{2W}{\rho C_D A}}, \tag{8}$$

where W is the combined weight of the skydiver and parachute, A is the effective cross-sectional area of the skydiver, and the density of air is $\rho = 1.225\,\text{kg/m}^3$. Solving for the drag coefficient C_D we obtain

$$C_D = \frac{2W}{\rho A v_T^2} = \frac{W}{qA}, \tag{9}$$

where q is the dynamic pressure corresponding to terminal velocity.

If a 72.6 kg skydiver carrying a 19 kg load (91.6 kg = 867 N) attains a terminal velocity of 49 m/s (in the belly-to-earth position) and has a cross-sectional area of 0.56 m^2, it follows from (9) that $C_D \approx (2 \times 867)/(1.225 \times 0.56 \times 49^2) = 1.05$.

Moreover, if our skydiver attains a terminal velocity of 67 m/s (in the head down or feet down position) and has a cross-sectional area of 0.2 m^2 in this position, it again follows from (9) that $C_D \approx (2 \times 867)/(1.225 \times 0.2 \times 67^2) = 1.57$. Actually, even if the skydiver could maintain the head down or feet down position over a long period of time, his rate of descent would continually slow due to the increasing density of air at lower altitudes.

In the 1960s, a 72.6 kg beginner sport parachutist might have used a circular parachute with a canopy area of 74.8 m^2, and would have carried about a 22.7 kg load (95.3 kg = 934 N) [**15**]. The parachute would have had $C_D \approx 0.8$. It follows from (8) that the terminal velocity for the parachutist is approximately $[(2 \times 934)/(1.225 \times 0.8 \times 74.7)]^{1/2} = 5.1$ m/s. Many measurements have confirmed that this prediction is quite a close approximation to the actual terminal velocity.

The sport parachutes used today bear little resemblance to the old classical round canopies, although the latter are still preferred by the military. The military's round canopies also have a relatively small area, which results in much harder landings than with modern sport canopies. Today, nearly all jumpers use either *square* (actually rectangular) or elliptical canopies, made from a non-porous material. When open, these canopies act like an airplane wing or an airfoil, and generate lift throughout the flight; they do not work by drag alone and are more like gliders than umbrellas. In addition, these modern square or elliptical canopies actually have *brakes* that the parachutist can apply close to the ground to achieve a gentle landing. Because of the lift that these canopies generate, their motion can not be modeled solely by the simple v^2 drag force model with the force parallel to motion.

However, we can obtain a reasonable model of a modern parachute by adding an extra term to (3) corresponding to the lift generated by the canopy. These are the same equations that are used to model flight of an unpowered airplane (a glider) or re-entry of the space shuttle into the earth's atmosphere. The force due to lift, F_L, is proportional to the square of the velocity, but now it is important to consider the horizontal component of motion−thus the new model is necessarily two dimensional and (3) is replaced by a pair of coupled nonlinear equations [**7**].

It is convenient to work in a special (rotating) coordinate system centered on the center of the earth. Letting V denote the tangential component of velocity for the unpowered aircraft, the equations of motion are

$$m\frac{dV}{dt} = -F_D - W\sin\theta, \qquad m\frac{V^2}{r_E} = -F_L + W\cos\theta \tag{10}$$

where θ denotes the *climb angle*, r_E is the radial distance of the aircraft to the center of the earth (which we approximate by the radius of the earth),

$$F_L = \tfrac{1}{2}C_L\,\rho A V^2, \qquad F_D = \tfrac{1}{2}C_D\,\rho A V^2,$$

C_L is the coefficient of lift, and C_D is the coefficient of drag (see Figure 2).

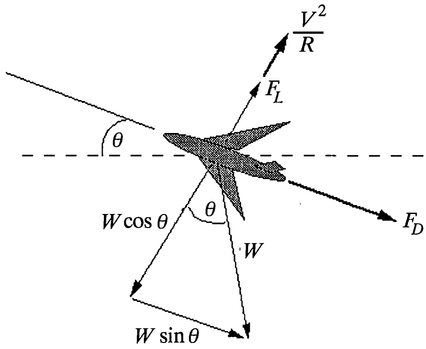

Figure 2. Forces on an unpowered aircraft

In general, even for a parachute, the equations in (10) do not have a closed form solution. However, there exists a closed form solution in one remarkable case that models re-entry of the space shuttle into the earth 's atmosphere. We discuss this example in Section 5.

5. RE-ENTRY OF THE SPACE SHUTTLE. The following model provides a reasonably accurate model for a *lifting body*, such as the space shuttle on re-entry into the atmosphere, with a closed form solution. This remarkable example should be much better known to mathematicians and can easily be presented in a first course on differential equations.

During much of the time during the space shuttle's re-entry, its velocity is approximately perpendicular to a line connecting the shuttle to the center of the earth, although at some instants the angle is quite large. In this model we assume that this is the case for all time. It then follows from (10), using $\theta = 0$, that the tangential velocity V of the shuttle satisfies

$$m\frac{dV}{dt} = -F_D, \qquad mV^2/r_E = -F_L + W, \tag{11}$$

where F_L = lift force = $C_L \rho V^2 A/2$, F_D = drag force = $C_D \rho V^2 A/2$, and r_E = radius of the earth.

For the space shuttle, it is reasonable to assume that $C_L \approx 0.5$, $C_D \approx 0.5$, $A \approx 372\, m^2$, and $W/(AC_L) \approx 100$. Over the flight envelope of the space shuttle, the quantity $F_L/F_D = C_L/C_D$ varies from about 1.0 to 1.8, and at high speeds it is roughly 1.0; for this simple example we approximate it by the constant 1.0.

We can rewrite (11) as

$$\frac{F_L}{W} = 1 - \left(\frac{V}{V_C}\right)^2 \quad \text{and} \quad \frac{F_D}{W} = -\frac{dV}{dt}\Big/ g,$$

where $V_C = \sqrt{g r_E}$. Dividing these equations gives the single equation

$$\frac{F_D}{F_L}\left(1 - \left(\frac{V}{V_C}\right)^2\right) = -\frac{dV}{dt}\Big/ g.$$

Since $F_D/F_L = C_D/C_L$, we obtain the separable equation

$$\frac{-\dfrac{dV}{V_c}}{1-\left(\dfrac{V^2}{V_c^2}\right)^2} = \frac{C_D g}{C_L V_c}\,dt,$$

which can be integrated to yield the closed form solution

$$V(t) = V_C \tanh\left(\frac{-C_D g}{C_L V_C}t + \operatorname{arctanh}\left(\frac{V_0}{V_C}\right)\right)$$

$$= V_C \tanh\left(\frac{-C_D g}{C_L\sqrt{g r_E}}t + \operatorname{arctanh}\left(\frac{V_0}{\sqrt{g r_E}}\right)\right), \tag{12}$$

where $V(0) = V_0$. Actual space shuttle flight test data **[5]** show that the velocity predicted by this simple model is reasonably accurate even though it is based on many simplifying assumptions.

One can use (12) to estimate the maximum acceleration experienced by the space shuttle upon re-entry.

6. PROOF OF THE PYTHAGOREAN THEOREM USING DIMENSIONAL ANALYSIS. We follow [**1**, p. 49] and give an insightful application of dimensional analysis to prove the Pythagorean theorem.

The area A of a right triangle is determined by its hypotenuse c and, for definiteness, the lesser of the acute angles ϕ: $A = f(c, \phi)$. Since the units of area are the square of units of length, dimensional analysis gives $A = c^2 g(\phi)$. The altitude perpendicular to the hypotenuse (see Figure 3) divides the basic triangle into two right triangles that are similar to it, and whose hypotenuses are the sides a and b of the original triangle. Their areas are $A_1 = a^2 g(\phi)$ and $A_2 = b^2 g(\phi)$. But $A = A_1 + A_2$, and thus $c^2 g(\phi) = a^2 g(\phi) + b^2 g(\phi)$. Hence $a^2 + b^2 = c^2$. ■

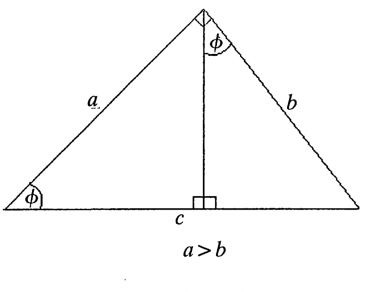

Figure 3. Right triangle

7. CONCLUSION. We have discussed models of motion for objects with *small* Reynolds numbers ($R < 1$) and for *large* Reynolds numbers ($R > \approx 100$). It is quite difficult to model the motion of most objects with Reynolds numbers in the intermediate range. The models we have discussed are quite popular with students and impress upon them, early in a differential equations course, the power of differential equations to model non-trivial physical phenomena. We applaud the trend in the new generation of calculus and differential equations texts to discuss more physical and biological models, and to make model building a major focus of the course. However, textbook writers and instructors should strive to present models based on correct physical or biological principles.

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REFERENCES

1. G. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press, 1996.
2. G. Birkhoff, *Hydrodynamics*, Princeton University Press, 1960.
3. J. Boyling, A Short Proof of the Pi Theorem of Dimensional Analysis, *Z. Angew. Math. Phys.* **30** (1979) 531–533.
4. P. Brancazio, *Sport Science*, Simon and Shuster, 1984.
5. H. Compton, et al., *Shuttle (STS − 1) Entry Trajectory Reconstruction*, AIAA Paper No. 81-2459 (1981).
6. M. Denny, *Air and Water: The Biology and Physics of Life's Media*, Princeton University Press, 1993.
7. W. L. Hankey, *Re-Entry Aerodynamics*, IAA Press, 1988.
8. S. Hoerner, *Fluid-Dynamic Drag*, 2nd edition, published by author, 1958, available by calling (206) 576–3997.
9. D. Hughes-Hallet, A. Gleason, et al., *Calculus*, Wiley, 1994.
10. T. Knacke, *The Parachute Recovery Systems Design Manual*, Para Publishing, 1992.
11. L. Landau and E. Lifshitz, *Fluid Mechanics*, 2nd edition, Butterworth and Heinemann, 1987.
12. B. McCormick, *Aerodynamics, Aeronautics, and Flight Mechanics*, Wiley, 1995.
13. R. D. Mehta, *Aerodynamics of Sports Balls* Annual Review of Fluid Mechanics **17** (1985), Annual Reviews Inc.
14. W. Meyer, *Concepts of Mathematical Modeling*, McGraw-Hill, 1984.
15. D. Poynter, *The Parachute Manual*, 3rd edition, Para Publishing, 1991.
16. D. Poynter, *The Parachute Manual, Volume II*, Para Publishing, 1991.
17. I. Proudman and J. Pearson, Expansions at Small Reynolds Numbers for the Flow Past a Sphere and a Circular Cylinder, *J. Fluid Mechanics* **2** (1957) 237–263.
18. R. S. Shevell, *Fundamentals of Flight*, Prentice-Hall, 1983.
19. P. A. Thompson, *Compressible Fluid Dynamics*, McGraw-Hill, 1972.
20. S. Vogel, *Life in Moving Fluids: The Physical Biology of Flow*, second edition, Princeton University Press, 1994.
21. R. von Mises and K. O. Fredrichs, *Fluid Dynamics*, Springer-Verlag, 1971.
21. F. M. White, *Viscous Fluid Flow*, 2nd edition, McGraw-Hill, 1991.

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