A Simple Proof of Rankin's Campanological Theorem

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Change ringing is the traditional English method of ringing church bells. The basic idea is to ring a set of bells in all possible orders (the changes) with no repetition until the initial position recurs. The sequence of changes is usually grouped into blocks, known as 'leads,' of a standard form, and one considers the sequence consisting of the last change in each lead (the 'lead ends'). Each lead end is obtained from the previous one by a permutation depending on the type of lead, and one tries to choose the sequence of leads so that all possible changes occur.

The mathematical problem involved in doing this can be formulated more generally as follows. Given a finite group G with a set of generators, E, one attempts to enumerate the elements of G as $x_1, \dots, x_n$ (with $n = |G|$) in such a way that for each $i, x_{i+1} = x_i e_i$ for some e_i in E (including $x_1 = x_n e_n$). Many explicit solutions have been given in particular cases, often by quite ingenious methods **[5]**, **[6]**, but few general results seem to be known about the possibility of constructing such a sequence. Aside from the obvious requirement that E generates G, the only necessary condition known to me is a theorem of Rankin **[4]**, which generalizes an earlier result of W. H. Thompson for a special case. This theorem asserts that if $E = \{a, b\}$ has at most 2 elements and if $c = ab^{-1}$ has odd order, then $|G: \langle a \rangle|$ and $|G: \langle b \rangle|$ must be odd. In fact, Rankin proved a more general result in which E is not required to generate G.

By a *cyclically ordered set* I mean a sequence $x_1, \dots, x_n$ of distinct elements, two such sequences $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ being regarded as the same if they differ by a cyclic permutation, i.e., $m = n$ and $y_i = x_{i+k}$ for some fixed k (indices being taken mod n). If G is a finite group and E is a subset of G, an *E-cycle* in G is a cyclically ordered subset $x_1, \dots, x_n$ of G such that the ratios $x_i^{-1} x_{i+1}$ all lie in E (including $x_n^{-1} x_1$).

Theorem 1 [4]. *Let $E = \{a, b\}$ be a subset of a finite group G. Suppose that G has a partition into r disjoint E-cycles. If $c = ab^{-1}$ has odd order, then $r \equiv |G: \langle a \rangle| \equiv |G: \langle b \rangle| \bmod 2$.*

Applications to change ringing may be found in **[4]**. Some history of Thompson's work is given in **[2]** and **[1]**. Our objective is to give a very simple proof of the theorem. With no more effort we can actually prove a somewhat more general result. Let X be a finite set and let E be a set of permutations of X. An *E-cycle in X* is a cyclically ordered subset $x_1, \dots, x_n$ of X such that for each $i = 1, \dots, n$, we have $x_{i+1} = \alpha_i(x_i)$ for some α_i in E. As always, the indices are taken mod n so that $x_1 = \alpha_n(x_n)$ also.

Theorem 2. *Let $E = \{\alpha, \beta\}$ where α and β are permutations of a finite set X having k and l cycles, respectively. Suppose X has a partition into r disjoint E-cycles. If $\gamma = \beta^{-1}\alpha$ has odd order, then $r \equiv k \equiv l \bmod 2$.*

Theorem 1 is an immediate consequence of Theorem 2. We let $X = G$ and define α and β to be right multiplication by a and b, i.e., $\alpha(x) = xa$ and $\beta(x) = xb$. Then $\gamma(x) = xc$ so γ has the same order as c and the cycles of α and β are just the left cosets of the subgroups $\langle a \rangle$ and $\langle b \rangle$. Therefore $k = |G: \langle a \rangle|$ and $l = |G: \langle b \rangle|$.

Remark. There are two obvious partitions of X into E-cycles, namely the cycles of α and the cycles of β. The point of Theorem 2 is thus that the parity of r is the same for all partitions into E-cycles if γ has odd order.

For the proof, observe that there is a 1-1 correspondence between partitions of X into disjoint cyclic subsets and permutations π of X, the cyclic subsets being the cycles of π. These cycles are E-cycles if and only if for each x in X we have $\pi(x) = \alpha_x(x)$ for some α_x in E. The parity of the number of cycles is related to the sign of π by the following fact.

Lemma 3 [3, App. A]. *Let π be a permutation of n elements having r cycles. Then* $\operatorname{sgn}(\pi) = (-1)^{n+r}$.

In fact, if π has p even cycles and q odd cycles, then $\operatorname{sgn}(\pi) = (-1)^p$, $r = p + q$, and $n \equiv q \bmod 2$.

In the situation of Theorem 2, let $P = \{x \in X | \pi(x) = \alpha(x)\}$ and $Q = \{x \in X | \pi(x) = \beta(x)\}$. Then $X = P \cup Q$. Let $\tau = \beta^{-1}\pi$. Then τ acts as the identity on Q, and $P - Q = X - Q$ is stable under τ, which clearly agrees with γ on it. So $\tau|P = \gamma|P$ and $\tau|Q = 1$. Therefore τ has odd order since γ does, so $\operatorname{sgn}(\tau) = 1$. It follows that $\operatorname{sgn}(\pi) = \operatorname{sgn}(\beta)$ and Lemma 3 shows that $r \equiv l \bmod 2$. Similarly, $r \equiv k \bmod 2$. ∎

Remark. We can also get some information if γ is not assumed to have odd order. Note that $P \cap Q = F$, the set of fixed points of γ. Since P and Q are stable under γ, they are determined by their images $\bar{P} = P/\langle \gamma \rangle$ and $\bar{Q} = Q/\langle \gamma \rangle$ in $\bar{X} = X/\langle \gamma \rangle$. Lemma 3 shows that $\operatorname{sgn}(\tau) = (-1)^d$, where $d = |P| + |\bar{P}|$. Since $\operatorname{sgn}(\pi) = \operatorname{sgn}(\beta)\operatorname{sgn}(\tau)$, we see that in all cases $r \equiv l + |P| + |\bar{P}| \bmod 2$. Similarly, $r \equiv k + |Q| + |\bar{Q}| \bmod 2$. In the situation of Theorem 1, $|P| = |\langle c \rangle| \cdot |\bar{P}|$ so if c has even order we get $r \equiv l + |\bar{P}| \bmod 2$ and similarly $r \equiv k + |\bar{Q}| \bmod 2$. It is also easy to see that, in the situation of Theorem 2, the possible partitions of X into E-cycles are in $1 - 1$ correspondence with subsets $\bar{P}$ of $\bar{X}$ that contain $\bar{F} = F$: We let $\bar{Q} = (\bar{X} - \bar{P}) \cup F$, let P and Q be the inverse images of $\bar{P}$ and $\bar{Q}$ in X, and define π to be α on P and β on Q.

Examples. The fact that P and Q are stable under γ was Thompson's key observation on which all proofs of the theorem are based. It has no analogue if E has more than 2 elements and it is not at all clear whether there is any analogue to Rankin's theorem for this case. One might guess that a similar conclusion holds if $E = \{e_1, \dots, e_k\}$ and we assume that all the elements $e_i e_j^{-1}$ have odd order. However, this is not the case. We give two examples, one having all indices $|G: \langle x \rangle|$ odd for $x \in E$ and one having all these indices even. Following the usual convention [2], an E-cycle $x_1, \dots, x_k$ having $x_{i+1} = x_i e_i$ with e_i in E and with $x_1 = 1$ is described by writing the word $e_1 e_2 \cdots e_k = 1$. If $x_1 \neq 1$, I write $x_1 \circ e_1 e_2 \cdots e_k = x_1$ instead.

(1) Let $G = S_3$ (the symmetric group) and let $E = \{a, b, c\}$ be the set of elements of order 2. It is well known [2] that there is a partition with $r = 1$, namely, $(ab)^3 = 1$. But there is also one with $r = 2$, namely, $abac = 1$ and $b \circ a^2 = b$.

(2) Let $G = A_4$ (the alternating group) and let $E = \{a, b, c\}$ with $a = (12)(34)$, $b = (123)$, and $c = (234)$. The coset decompositions for $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$ all have r even but there is also a partition with $r = 1$, namely, $(c^2ac^2b)^2 = 1$.

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REFERENCES

1. D. J. Dickinson, On Fletcher's paper "Campanological Groups," *Amer. Math. Monthly* **64** (1957) 331–332.
2. T. J. Fletcher, Campanological groups, *Amer. Math. Monthly* **63** (1956) 619–626.
3. A. G. Kurosh, *The Theory of Groups*, vol. 1, Chelsea, New York, 1955.
4. R. A. Rankin, A campanological problem in group theory, *Proc. Camb. Phil. Soc.* **44** (1948) 17–25.
5. A. T. White, Ringing the cosets, *Amer. Math. Monthly* **94** (1987) 721–746.
6. A. T. White, Fabian Stedman: The first group theorist?, *Amer. Math. Monthly* **103** (1996) 771–778.

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