n-Ellipses and the Minimum Distance Sum Problem

Junpei Sekino

1. INTRODUCTION. When we hear *ellipse*, we might think of planetary orbits or rooms with magical acoustic properties. In all such examples of ellipses in nature, the *foci* play a distinguished role. Our goal is to consider a natural class of generalized ellipses given by an arbitrary number of foci. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be n distinct points in the plane, and let k be a positive constant. By the *n-ellipse with foci* $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ and the *distance sum* k, we mean the level set of the *distance sum function*

$$f(\mathbf{r}) = \sum_{i=1}^{n} |\mathbf{r} - \mathbf{c}_i| \tag{1}$$

at the level $f(\mathbf{r}) = k$. We show that if k is sufficiently large (as explained in Theorem 6), then the n-ellipse is a piecewise smooth Jordan curve whose interior is convex; it is nonsmooth only where it contains a focus. The n-ellipses have diverse shapes that include curves resembling contours of eggs, lemons, pears, and even human faces, symmetric or asymmetric. Circles are 1-ellipses and ordinary ellipses are 2-ellipses.

Surveying a family of n-ellipses given by a set of n foci on a computer screen reveals not only a variety of continuously changing contours but also the existence of a *center*, which is the *unique* point that minimizes the distance sum function; see Figure 1. It is interesting to explore the general behavior of the distance sum function but it is also important to examine its local properties near the center. The latter leads us to the critical points of the distance sum function and to

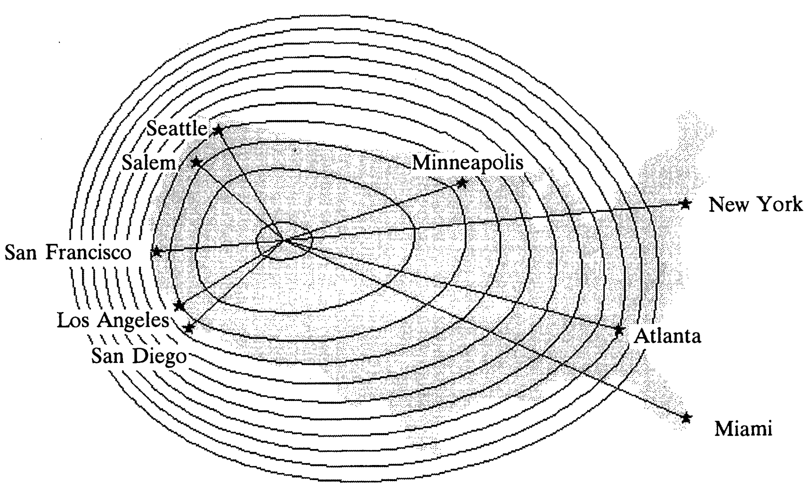

Figure 1. A family of 9-ellipses and a center

geometrically intriguing properties. We show that a critical point is nondegenerate if and only if the foci are noncollinear. This theorem, which places circles and ordinary ellipses in a minor degenerate league, shows that the geometrically and analytically appealing case arises when the foci do not lie in a straight line. The existence of a center under this condition follows quickly.

One outcome of our exploration is a mathematical model for certain optimization problems, and the final section of this paper lists a few examples that can be solved by a contour map and the properties of critical points. These problems may be given to students in a second year calculus or optimization class as a challenging project.

2. CONTOUR PLOTTING AND EXAMPLES. Sherlock Holmes said, "It is a capital crime to theorize before one has data." To gather data and develop our intuition, we used a contour plotter to draw a family of n-ellipses on the computer screen and display an approximate location of the center. We wrote a program to perform the task **[4]**; commercial contouring routines are available in computer algebra systems such as *Maple* and *Mathematica*.

Example 1. Figure 1 shows a family of 9-ellipses and its center, which is related to a project in the final section. It shows that an n-ellipse need not contain all foci (indicated by stars) in its interior when $n > 2$.

Example 2. As Figure 2 shows, a 3-ellipse generated by the vertices of an isosceles triangle can resemble the familiar section of an egg. The vertices of a rectangle generate 4-ellipses that resemble an ordinary ellipse, and so does a finite set of equally spaced collinear points.

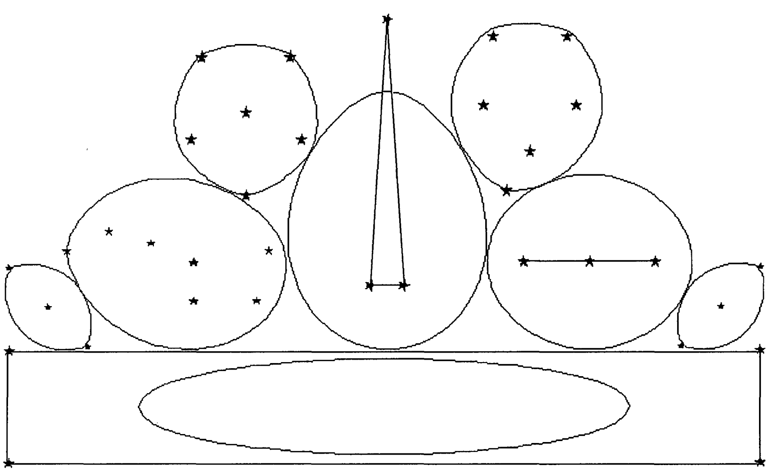

Figure 2. Assorted n-ellipses

3. PRELIMINARY RESULTS. Consider the simple distance function

$$g(\mathbf{r}) = |\mathbf{r} - \mathbf{c}| = \sqrt{(x-p)^2 + (y-q)^2} \tag{2}$$

centered at a point $\mathbf{c} = (p, q)$ where $\mathbf{r} = (x, y)$. The domain of definition of g is the entire plane, and g is a C^∞-function on the plane with the focus $\mathbf{c}$ removed. A straightforward calculation gives $\nabla|\mathbf{r}-\mathbf{c}| = (\mathbf{r}-\mathbf{c})/|\mathbf{r}-\mathbf{c}|$ for $\mathbf{r} \neq \mathbf{c}$, so $\nabla g(\mathbf{r})$ is the (unit) direction vector from $\mathbf{c}$ to $\mathbf{r}$. We shall employ the convenient notation $\nabla|\mathbf{b}-\mathbf{a}|$ for the direction from a point $\mathbf{a}$ to another point $\mathbf{b}$. By the linearity of ∇, we have:

Theorem 1. *The distance sum function f in* (1) *is C^∞ on the plane with the foci removed, and*

$$\nabla f(\mathbf{r}) = \sum_{i=1}^{n} \nabla|\mathbf{r}-\mathbf{c}_i| = \sum_{i=1}^{n} \frac{\mathbf{r}-\mathbf{c}_i}{|\mathbf{r}-\mathbf{c}_i|}.$$

Thus, ∇f is a gradient field on the plane with holes at the foci; we draw the gradient $\nabla f(\mathbf{r})$ by placing its tail at $\mathbf{r}$. According to Theorem 1, each $\nabla f(\mathbf{r})$ is completely determined by the directions $\nabla|\mathbf{r}-\mathbf{c}_i|$, $i = 1, 2, \ldots, n$, which we call the *direction components* of $\nabla f(\mathbf{r})$. Figure 3 illustrates a 2-ellipse and a 3-ellipse with gradients and their direction components (bold arrows) and the tangent lines perpendicular to the respective gradients. The 2-ellipse shows that the line segments $\mathbf{rc}_1$ and $\mathbf{rc}_2$ make equal angles with the tangent line. This "ball bouncing property" of an ordinary ellipse from one focus to another off the wall is not available in n-ellipses if $n > 2$.

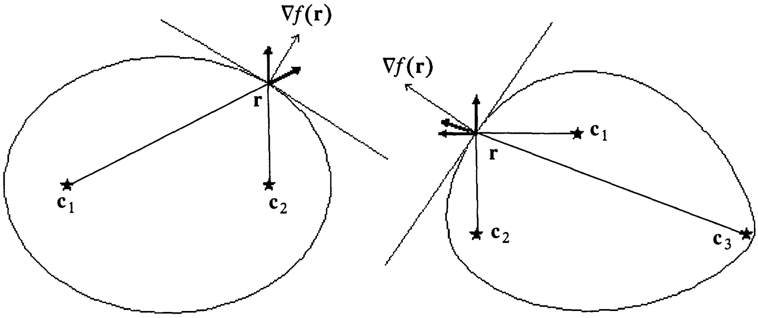

Figure 3. A 2-ellipse and a 3-ellipse

The next lemma implies that every vertical section of the distance sum function f has *at most* one "valley" corresponding to a local minimum and no "hill" that corresponds to a local maximum.

Lemma 1. *Let L be any line, and parametrize L by $\mathbf{r}(t) = \mathbf{d}t + \mathbf{b}$ where $\mathbf{d}$ is a unit vector. Then $f(\mathbf{r}(t))$ is continuous and the directional derivative of f along L is piecewise continuous and monotone increasing* (*i.e.*, *nondecreasing*).

Proof: First, consider the distance function g in (2). The directional derivative $\nabla g(\mathbf{r}(t)) \cdot \mathbf{d}$ is the scalar projection of the direction vector $\nabla|\mathbf{r}(t)-\mathbf{c}|$ onto $\mathbf{d}$ and therefore it is increasing if $\mathbf{c} \notin L$ (see Figure 4). Similarly, if $\mathbf{c} \in L$, $\nabla g(\mathbf{r}(t)) \cdot \mathbf{d}$ is a monotone increasing step function that is undefined when $\mathbf{r}(t) = \mathbf{c}$. Since $f(\mathbf{r}(t))$ is a finite sum of functions of the form $g(\mathbf{r}(t))$, $f(\mathbf{r}(t))$ is continuous and $\nabla f(\mathbf{r}(t)) \cdot \mathbf{d}$ is monotone increasing. The directional derivative does not exist when $\mathbf{r}(t)$ coincides with a focus of f. ■

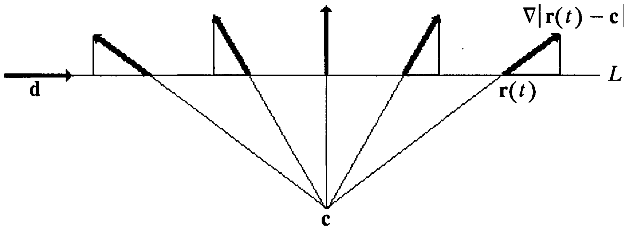

Figure 4. The directional derivative of g along L

Theorem 2. *The function f in* (1) *has a global minimum.*

Proof: Choose a compact disk D containing all the foci in its interior, and let C be the boundary of D. Then f attains a global minimum value M on D at some point $\mathbf{s} \in D$. Theorem 1 ensures that $\nabla f(\mathbf{r}) \neq \mathbf{0}$ for all $\mathbf{r} \in C$, and therefore, $\mathbf{s}$ belongs to the interior of D. Let L be any ray in the plane emanating from $\mathbf{s}$, and let $\mathbf{b}$ be the intersection of L and C. Parametrize L by $\mathbf{r}(t) = \mathbf{d}t + \mathbf{s}$, $t \geq 0$, where $\mathbf{d} = \nabla|\mathbf{b} - \mathbf{s}|$. Then Lemma 1 and $f(\mathbf{s}) = M$ imply $\nabla f(\mathbf{b}) \cdot \mathbf{d} \geq 0$. Appealing to Lemma 1 once again, therefore, we have $M = f(\mathbf{s}) \leq f(\mathbf{r}(t))$ for all $t \geq 0$. This proves that M is the global minimum value of f on L, and the theorem follows. Note that M need not be achieved at a unique point. ∎

4. CRITICAL POINTS. By a *critical point* (abbreviated *CP*), we mean a point $\mathbf{r}$ where $\nabla f(\mathbf{r}) = \mathbf{0}$. This includes the assumption that $\mathbf{r}$ is not a focus. We say that a CP is *with n* foci if f has exactly n foci.

Theorem 1 ensures that $\mathbf{r}$ is a CP if and only if

$$\sum_{i=1}^{n} \nabla|\mathbf{r} - \mathbf{c}_i| = \sum_{i=1}^{n} \frac{\mathbf{r} - \mathbf{c}_i}{|\mathbf{r} - \mathbf{c}_i|} = \mathbf{0}. \tag{3}$$

While this formula discourages algebraic approaches to a solution, it gives a striking geometric property of a CP: *The directions from the foci add up to zero precisely at a CP, whence a CP depends only upon the directions from the foci but not the distances.* Figure 5 stresses this point and shows distinct sets of foci (one set for

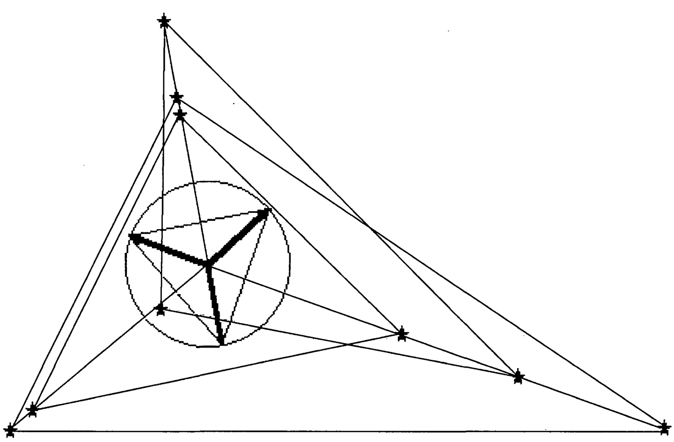

Figure 5. Relationship between the foci and a CP

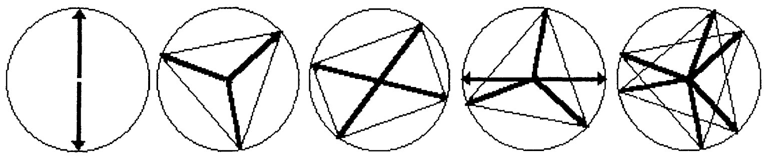

Figure 6. Examples of CP-patterns with 2, 3, 4, 5, and 6 foci

each triangle) that have the same CP (the center of the unit circle). The circular part of Figure 5, which we call a *CP-pattern*, consists of a CP and the direction components (bold arrows) of $\nabla f(\mathbf{r})$ whose sum vanishes at the CP. Note that the *negative* of each direction component $\nabla|\mathbf{r} - \mathbf{c}_i|$ points toward the focus $\mathbf{c}_i$. Figure 6 illustrates a few more CP-patterns. The first CP-pattern of Figure 6 describes the only way ∇f can vanish when there are exactly two foci, and therefore $\mathbf{r}$ is a CP with two foci if and only if $\mathbf{r}$ is strictly between the foci; the second CP-pattern shows that $\mathbf{r}$ is a CP with three foci if and only if the angle between any pair of the direction components is 120°; the third pattern shows that $\mathbf{r}$ is a CP with four noncollinear foci if and only if $\mathbf{r}$ is the center of the quadrilateral formed by the foci. The idea of locating CPs by means of the CP-patterns becomes elusive, however, if $n \geq 5$. The fourth pattern shows a CP with five foci, and the rotations of the horizontal vectors bound together yield infinitely many distinct CP-patterns with five foci. In addition, a regular pentagon gives yet another CP-pattern with five foci, and there are others that match none of the above. The fifth pattern is just one of the infinitely many CP-patterns with six foci.

The implication from the CP-pattern with two foci just observed can be extended easily to the following:

Theorem 3. *Suppose the foci of an n-ellipse are collinear. If n is even then a point* $\mathbf{r}$ *is a CP if and only if* $\mathbf{r}$ *lies strictly between the middle two foci. If n is odd, no CPs exist.*

According to the theorem, the distance sum function f can have *no* CPs or *infinitely many* CPs, and this raises the question: If the foci are noncollinear, how many CPs can f have? The second and third patterns in Figure 6 indicate that f can have at most one CP if the number of foci is 3 or 4, but it is not clear what happens for more foci. To settle this question, therefore, we take an analytic route. We say that a CP $\mathbf{r}$ is *degenerate* if

$$Hf(\mathbf{r}) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y\,\partial x} \\[2ex] \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix},$$

the *Hessian matrix* of f at $\mathbf{r}$, is singular.

Theorem 4. *A CP of an n-ellipse is nondegenerate if and only if the foci are noncollinear. Furthermore, every nondegenerate CP is a local minimum of the distance sum function.*

Proof: Let $\mathbf{r} = (x, y)$ and $\mathbf{c}_i = (p_i, q_i)$. To avoid cluttered formulas, let

$$X_i = x - p_i, \quad Y_i = y - q_i, \quad \text{and} \quad g_i = g_i(\mathbf{r}) = |\mathbf{r} - \mathbf{c}_i| = \sqrt{X_i^2 + Y_i^2}.$$

Then

$$\frac{\partial g_i}{\partial x} = \frac{X_i}{g_i}, \qquad \frac{\partial g_i}{\partial y} = \frac{Y_i}{g_i},$$

and

$$\nabla f(\mathbf{r}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\sum \frac{\partial g_i}{\partial x}, \sum \frac{\partial g_i}{\partial y}\right) = \left(\sum \frac{X_i}{g_i}, \sum \frac{Y_i}{g_i}\right),$$

where each summation is taken over $i = 1, 2, \ldots, n$. Hence,

$$\frac{\partial^2 f}{\partial x^2} = \sum \frac{\partial}{\partial x}\frac{X_i}{g_i} = \sum \frac{\frac{\partial X_i}{\partial x} g_i - X_i \frac{\partial g_i}{\partial x}}{g_i^2} = \sum \frac{g_i - \frac{X_i^2}{g_i}}{g_i^2} = \sum \frac{g_i^2 - X_i^2}{g_i^3} = \sum \frac{Y_i^2}{g_i^3},$$

and similarly,

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} = -\sum \frac{X_i Y_i}{g_i^3}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \sum \frac{X_i^2}{g_i^3}.$$

Setting $A_i = X_i / g_i^{3/2}$ and $B_i = Y_i / g_i^{3/2}$, the Hessian matrix of f can be written as

$$Hf(\mathbf{r}) = \begin{bmatrix} \sum B_i^2 & -\sum A_i B_i \\ -\sum A_i B_i & \sum A_i^2 \end{bmatrix}.$$

Therefore,

$$\det Hf(\mathbf{r}) = \left(\sum A_i^2\right)\left(\sum B_i^2\right) - \left(\sum A_i B_i\right)^2 = |A|^2 |B|^2 - (A \cdot B)^2$$

where $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$. It follows from the Cauchy-Schwarz inequality that $\det Hf(\mathbf{r}) \geq 0$, and $\det Hf(\mathbf{r}) = 0$ if and only if A and B are linearly dependent. The rest follows from the second derivative test. ∎

According to Theorem 4, the CPs with 3, 4, 5, and 6 foci in Figure 6 are all nondegenerate. Since nondegenerate CPs are isolated [**3**, p. 8], Theorem 4 and Lemma 1 imply

Corollary 1. *If the foci of an n-ellipse are noncollinear, the distance sum function f has at most one CP, i.e., a CP is unique if it exists.*

We say that a point $\mathbf{s}$ is the *center* of the distance sum function f if $\mathbf{s}$ is the *unique* point at which f attains a global minimum. We now show that f has a center unless the foci are collinear and the number of foci is even.

Theorem 5. *Let an n-ellipse be given. (A) Suppose the foci are noncollinear. If a CP exists then it is the center; otherwise, one of the foci coincides with the center. (B) Suppose the foci are collinear. If n is even, then f has no center, and instead, f attains a global minimum at* $\mathbf{r}$ *if and only if* $\mathbf{r}$ *lies in the closed line segment joining the middle two foci; if n is odd, then the middle focus is the center of f.*

Proof: Let S be the set consisting of all CPs and foci and let M be the global minimum value of f. Since every local minimum of f is either a CP or a point at which f is not differentiable, it follows that f attains the value M at some point $\mathbf{s} \in S$ but never at a point in the complement of S.

(A) S is finite by Corollary 1, and Lemma 1 guarantees uniqueness of the point $\mathbf{s}$ where $f(\mathbf{s}) = M$, i.e., $\mathbf{s}$ is the center of f. Now, if no CPs exist then $\mathbf{s}$ must be a focus; if there is a CP at $\mathbf{r}$ then Theorem 4 ensures that $f(\mathbf{r})$ is a local minimum; appealing to Lemma 1 again, we conclude that $\mathbf{r} = \mathbf{s}$.

(B) Suppose n is even, and let L be the line through the foci. Then $S \subset L$ by Theorem 3, and consequently the global minimum value of f over L coincides with M. Let $\mathbf{d}$ be a direction of L and suppose without loss of generality that the foci are lined up on L in the direction $\mathbf{d}$ as

$$\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_j, \mathbf{c}_{j+1}, \ldots, \mathbf{c}_{2m} \quad \text{so that} \quad \mathbf{d} = \nabla|\mathbf{c}_2 - \mathbf{c}_1|. \tag{4}$$

Parametrize L by $\mathbf{r}(t) = \mathbf{d}t + \mathbf{b}$, and given any index j, choose $t = t_j$ in such a way that $\mathbf{r}_j = \mathbf{r}(t_j)$ is strictly between $\mathbf{c}_j$ and $\mathbf{c}_{j+1}$ on L. Then $\mathbf{r}_j$ separates L into two rays R_1 and R_2 such that $\mathbf{c}_j \in R_1$, and therefore R_1 and R_2 contain exactly j and $2m - j$ foci, respectively. Theorem 1 and (4) ensure that

$$\begin{aligned}\nabla f(\mathbf{r}_j) &= \sum_{i=1}^{j} \nabla|\mathbf{r}_j - \mathbf{c}_i| + \sum_{i=j+1}^{2m} \nabla|\mathbf{r}_j - \mathbf{c}_i| \\ &= \sum_{i=1}^{j} \nabla|\mathbf{c}_2 - \mathbf{c}_1| + \sum_{i=j+1}^{2m} \nabla|\mathbf{c}_1 - \mathbf{c}_2| = \sum_{i=1}^{j} \mathbf{d} - \sum_{i=j+1}^{2m} \mathbf{d} = 2(j-m)\mathbf{d},\end{aligned}$$

and $\nabla f(\mathbf{r}_j) \cdot \mathbf{d} = 2(j - m)\,|\,\mathbf{d}\,|^2 = 2(j - m)$. Consequently, $d(f(\mathbf{r}(t)))/dt = \nabla f(\mathbf{r}) \cdot \mathbf{d} = 0$ if and only if $\mathbf{r}(t)$ is strictly between $\mathbf{c}_m$ and $\mathbf{c}_{m+1}$. Since $f(\mathbf{r}(t))$ is continuous, we now conclude that $f(\mathbf{r}(t)) = M$ if and only if $\mathbf{r}(t)$ lies in the closed line segment joining the middle two foci. A similar argument proves the second part of (B). ■

Although Theorem 5(A) guarantees the existence of a (unique) center under any noncollinear arrangement of the foci, it is still incomplete in the sense that it tells neither when a CP exists nor which focus is the center if no CPs exist. Theorem 5(A) can be strengthened as follows:

Suppose the foci are noncollinear and therefore a CP is unique whenever it exists (Corollary 1). From the second and third CP-patterns in Figure 6, we have:

(C) There is a unique CP with three foci if and only if the foci form a triangle whose interior angle never exceeds or equals 120°.

(D) There is a unique CP with four foci if and only if the foci form a convex quadrilateral whose interior angle never equals 180° .

If a CP with three foci fails to exist, which focus is the center? The answer is the focus that corresponds to the greatest interior angle of the triangle, which can be verified easily by checking the directional derivative of f on the line segments joining it and the other foci. The answer is the same for four foci. For more foci, CPs abound:

(E) There is a unique CP with n foci if the foci form a convex n-gon whose interior angle never equals 180°.

The sufficient condition in (E) is not necessary: We can easily build a counterexample using the CP-pattern with five foci in Figure 6. To justify (E), suppose $n = 5$

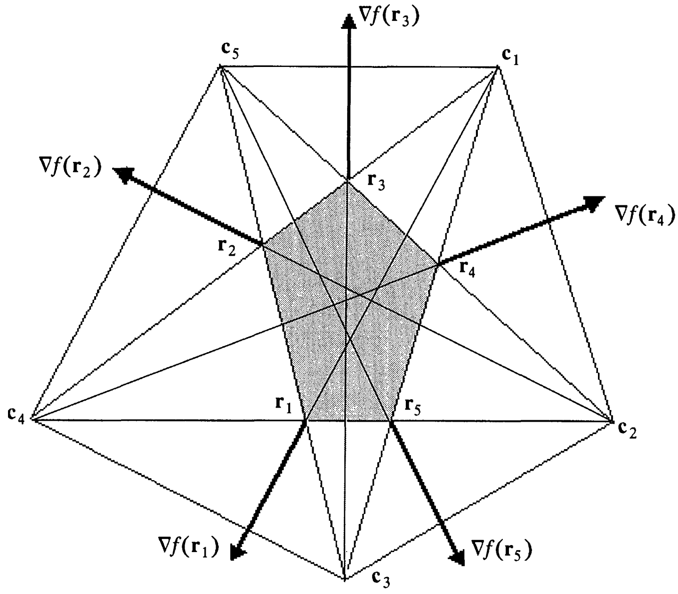

Figure 7. Is the center inside the little pentagon?

and consider the convex pentagon in Figure 7. Each $\mathbf{r}_i$ is a vertex of the smaller pentagon constructed by the "star-forming" diagonals. Let P be the smaller pentagon and let L be the closed line segment joining adjacent vertices of P, say $\mathbf{r}_1$ and $\mathbf{r}_2$. Each gradient $\nabla f(\mathbf{r}_i)$, $i = 1, 2$, is given by offsetting four of its direction components as shown in the picture, and as a result,

$$-\nabla f(\mathbf{r}_i) \text{ points into the interior of } P, \tag{5}$$

i.e., $-\nabla f(\mathbf{r}_i)$ has a positive projection on the inward unit normal to L at $\mathbf{r}_i$.

Parametrize L by $\mathbf{r}(t) = \mathbf{d}t + \mathbf{r}_1$, where $\mathbf{d} = \nabla |\mathbf{r}_2 - \mathbf{r}_1|$. Since L does not contain any focus, the directional derivative $\nabla f(\mathbf{r}(t)) \cdot \mathbf{d}$ is continuous, and Lemma 1 ensures that its negative

$$-\nabla f(\mathbf{r}(t)) \cdot \mathbf{d} = \cos \theta(\mathbf{r}(t))$$

is monotone decreasing, where $\theta(\mathbf{r}(t)) = \cos^{-1}(-\nabla f(\mathbf{r}(t)) \cdot \mathbf{d})$ is the angle between $-\nabla f(\mathbf{r}(t))$ and $\mathbf{d}$. Consequently, $\theta(\mathbf{r}(t))$ is monotone increasing from $\theta(\mathbf{r}_1)$ to $\theta(\mathbf{r}_2)$ where $0° < \theta(\mathbf{r}_1) < \theta(\mathbf{r}_2) < 180°$, and the vector field $-\nabla f$ points into the interior of P along the line segment L (except possibly at the CP if indeed L contains it). This implies that P contains the center, which in turn coincides with the CP of f (Theorem 5(A)).

If $n > 5$, $\nabla f(\mathbf{r}_i)$ is the sum of $n - 4$ direction components but the preceding argument carries over as the property (5) can be observed easily under the general circumstance.

Our final theorem concerns the general shape of n-ellipses.

Theorem 6. *Let M be the global minimum value of f. Every n-ellipse with the distance sum greater than M is a piecewise smooth Jordan curve and its interior is a nonempty convex set.*

Proof: Given $k > M$, let E be the n-ellipse with $f(\mathbf{r}) = k$. Let $S = f^{-1}[M, k]$ and let $\text{int}(S)$ denote the interior of S. Then $\text{int}(S) = f^{-1}[M, k)$, which is nonempty.

To show the convexity of int(S), let $\mathbf{r}_1, \mathbf{r}_2 \in \text{int}(S)$, and parametrize the line L through $\mathbf{r}_1, \mathbf{r}_2$ by $\mathbf{r}(t) = \mathbf{d}t + \mathbf{r}_1$. Then Lemma 1 ensures that $f(\mathbf{r}) \leq \max\{f(\mathbf{r}_1), f(\mathbf{r}_2)\} < k$, whenever $\mathbf{r}$ is in L between $\mathbf{r}_1$ and $\mathbf{r}_2$. This means that the portion of L between $\mathbf{r}_1$ and $\mathbf{r}_2$ is in int(S), so int(S) is convex. Next, we show that S is bounded, and therefore it is compact. Theorem 5 guarantees that there is a point $\mathbf{s}$ such that $f(\mathbf{s}) = M$. If $\mathbf{r} \in S$ then $|\mathbf{r} - \mathbf{s}| \leq |\mathbf{r} - \mathbf{c}_i| + |\mathbf{s} - \mathbf{c}_i|$ for each focus $\mathbf{c}_i$, so

$$n|\mathbf{r} - \mathbf{s}| \leq \sum_{i=1}^{n} |\mathbf{r} - \mathbf{c}_i| + \sum_{i=1}^{n} |\mathbf{s} - \mathbf{c}_i| = f(\mathbf{r}) + f(\mathbf{s}) \leq k + M.$$

This proves that S is bounded. Now Theorem 5 also guarantees that all CPs are in int(S) whenever they exist. Therefore the Implicit Function Theorem ensures that E is a piecewise smooth curve: If no focus lies in E, then E is smooth; otherwise, cusps may occur at the foci that connect the smooth segments given by the Implicit Function Theorem. The n-ellipse E bounds a convex and compact region S, whence it must be a Jordan curve whose interior coincides with int(S). ∎

5. STUDENT PROJECTS. The following problems are open-ended and solutions need not be unique. Your report must describe the logic behind your solutions and must explain how you used computers or calculators. Use the map provided (Figure 1 without the curves and lines).

A woman owns a trendy specialty-food store in each of the nine cities shown in the map.

(a) She plans to open another store in the United States. The new site is to be her headquarters and she wants the location to be the most convenient for frequent trips she makes by her private airplane to the nine cities. If she visits each city equally frequently directly from the headquarters, where should she locate the new store?

In the following, suppose you have solved the nine city problem and that the owner of the chain has built her headquarters.

(b) She wants to open two more stores including one in San Antonio, but she does not want to move her headquarters. What other city should she choose? What if "two" were replaced by "three"? "four"? "five"?

(c) If she wants to relocate the Atlanta store without moving the headquarters, where should she move it?

(d) If changing American eating habits force her to close two of the nine stores, but she does not want to relocate her headquarters, which two stores should she close?

(Possible Solutions) The contour plot of Figure 1, which is given by assigning a coordinate system on the map, points to a mountainous area near the Utah-Wyoming border: (a) Choose Salt Lake City, the largest city in the area. From the geometric interpretation of (3), we also have: (b) Boise, Idaho, which "counterbalances" San Antonio with respect to the center; (c) Choose a city on the line segment between Salt Lake City and Atlanta; Yes, moving the store from Atlanta to, say Denver, changes the contours but the center stays miraculously in the same area. (d) San Francisco and New York.

Inquisitive students may wish to extend the idea of n-ellipse to that of *weighted n-ellipse* given by the new function

$$f(\mathbf{r}) = \sum_{i=1}^{n} w_i |\mathbf{r} - \mathbf{c}_i|$$

with *weights* $w_i > 0$. The critical points $\mathbf{r}$ of f satisfy the equation $\sum_{i=1}^{n} w_i \nabla |\mathbf{r} - \mathbf{c}_i| = 0$. This causes the corresponding CP-patterns to become more complex, but as a trade-off, the weights provide us with optimization problems that are more realistic. The following example shows the effect of weights on the center:

(e) Because of the varying business size, the owner of the chain in (a) above put the following weights on the nine cities: Seattle = 2.5, Salem = 1, SF = 2.4, LA = 2.2, SD = 2.4, Miami = 2.2, Atlanta = 2, NY = 6, and Minneapolis = 2. This means, for example, that she visits NY six times as frequently as Salem, Oregon. Where should she locate her headquarters? Figure 8 shows that the center shifts

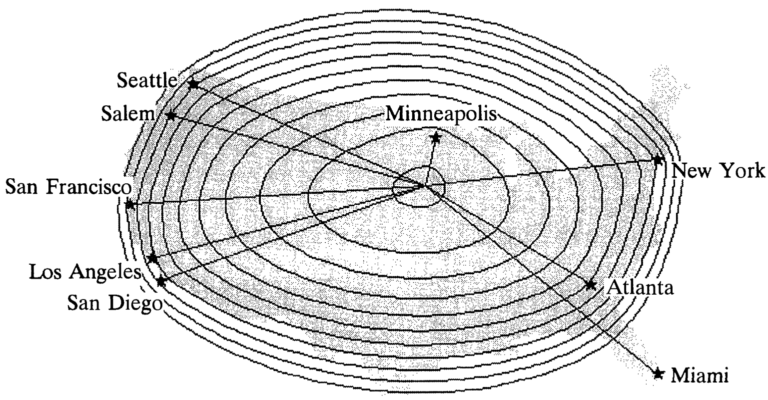

Figure 8. A family of weighted 9-ellipses

eastward to a point near Iowa City mainly due to the heavy weight placed on New York.

REFERENCES

1. W. Kaplan, *Advanced Calculus*, 4th ed., Addison Wesley, Reading, 1991.
2. J. E. Marsden and A. J. Tromba, *Vector Calculus*, 4th ed., Freeman, San Francisco, 1996.
3. J. W. Milnor, *Morse Theory*, Princeton University Press, Princeton, 1969.
4. J. Sekino, N. Liepins, and V. Morachevsky, Contour Plot for Smooth Surfaces, Proceedings of St. Petersburg State University (Astronomy/Mathematics/Mechanics Branch) **4** (22) (1994), 117–118.

JUNPEI SEKINO finished an undergraduate program in chemistry at Nihon University, Tokyo, and came to Oregon State University, where he was influenced by teachers/mentors Gordon W. Gilkey, Ze'ev B. Orzech, and J. Wolfgang Smith. After receiving a second bachelor's degree in art, he earned his Ph.D. in algebraic topology under Smith. He is professor at Willamette University, where he was once the Mortarboard Professor of the Semester. His hobbies include wild mushroom hunting, printmaking, and fractal plotting. His prizewinning web site "Sekino's Fractal Gallery" can be accessed via the home page of the MAA's Pacific Northwest Section (http://www.math.ubc.ca/ ~ cayf/Pacnortsect.html).
Willamette University, Salem, OR 97301
sekino@willamette.edu