THE EVOLUTION OF . . .

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The Literal Calculus of Viète and Descartes

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Translated from the Russian by Abe Shenitzer†

1. The contribution of François Viète (1540–1603). Viète tried to create a new science (he called it ars analytica, or analytic art) that would combine the rigor of the geometry of the ancients with the operativeness of algebra. This analytic art was to be powerful enough to leave no problem unsolved: nullum non problema solvere.

Viète set down the foundations of this new science in his *An introduction to the art of analysis* (In artem analyticem isagoge) of 1591.

In this treatise he created a literal calculus. In other words, he introduced the language of formulas into mathematics. Before him, literal notations were restricted to the unknown and its powers. Such notations were first introduced by Diophantus and were somewhat improved by mathematicians of the 15th and 16th centuries.

The first fundamentally new step after Diophantus was taken by Viète, who used literal notations for parameters as well as for the unknown. This enabled him to write equations and identities in general form. It is difficult to overestimate the importance of this step. Mathematical formulas are not just a compact language for recording theorems. After all, theorems can also be stated by means of words; for example, the formula

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

can be expressed by means of the phrase "the square of the sum of two quantities is equal to the square of the first quantity, plus the square of the second quantity, plus twice their product." Shorthand also has the virtue of brevity. What counts is that we can carry out operations on formulas in a purely mechanical manner and obtain in this way new formulas and relations. To do this we must observe three rules: 1) the rule of substitution; 2) the rule for removing parentheses; and 3) the rule for reduction of similar terms. For example, from formula (1) one can obtain in a purely mechanical manner, without reasoning, formulas for $(a+b+c)^2$, for $(a+b)^3$, and so on. In other words, literal calculus replaces some reasoning by mechanical computations. In Leibniz' words, literal calculus "relieves the imagination".

†*Translator's note.* The process of creating a literal algebra was begun by Diophantus and was completed by Viète and Descartes. The contribution of Diophantus was described in "The Birth of Literal Algebra", this MONTHLY 106 (1999) 260–271. Viète and Descartes, created the literal calculus we use today.

This article is practically all of Section 4, Chapter V, and Section 1, Chapter VI of [**1**].

We can hardly imagine mathematics without formulas, without a calculus. But it was such up until Viète's time. The importance of the step taken by Viète is so fundamental that we consider his reasoning in detail.

Viète adopted the basic principle of Greek geometry, according to which only homogeneous magnitudes can be added, subtracted, and can be in a ratio to one another. As he put it: "Homogena homogenei comparare." As a result of this principle, he divides magnitudes into "species": the 1st species consists of "lengths", i.e., of one-dimensional magnitudes. The product of two magnitudes of the 1st species belongs to the 2nd species, which consists of "plane magnitudes", or "squares", and so on.

In modern terms, the domain V of magnitudes considered by Viète can be described as follows:

$$V = \mathbf{R}_+^{(1)} \cup \mathbf{R}_+^{(2)} \cup \cdots \cup \mathbf{R}_+^{(k)} \cup \cdots,$$

where $\mathbf{R}_+^{(k)}$ is the domain of k-dimensional magnitudes, $k \in \mathbf{N}_+$. In each of the domains $\mathbf{R}_+^{(k)}$ we can carry out the operations of addition and of subtraction of a smaller magnitude from a larger one, and can form ratios of magnitudes. If $\alpha \in \mathbf{R}_+^{(k)}$ and $\beta \in \mathbf{R}_+^{(l)}$, then there is a magnitude $\gamma = \alpha\beta$ and $\gamma \in \mathbf{R}_+^{(k+l)}$. If $k > l$, then there exists a magnitude $\delta = \alpha : \beta$, and $\delta \in \mathbf{R}_+^{(k-l)}$.

After constructing this "ladder", Viète proposes to denote unknown magnitudes by vowels A, E, I, O... and known ones by consonants $B, C, D, \ldots$ Furthermore, to the right of the letter denoting a magnitude he places a symbol denoting its species. Thus if $B \in \mathbf{R}_+^{(2)}$, then he writes B plan (i.e., planum–plane), and if an unknown $A \in \mathbf{R}_+^{(2)}$, then he writes A quad (square). Similarly, magnitudes in $\mathbf{R}_+^{(3)}$ get the indices solid or cub and those in $\mathbf{R}_+^{(4)}$ get the indices plano-planum or quadrato-quadratum, and so on.

For addition and subtraction Viète adopts the cossist symbols $+$ and $-$ and introduces the symbol $=\!=$ for the absolute value of the difference of two numbers, thus $B =\!= D$ is the same as $|B - D|$. For multiplication he uses the word "in", A in B, and for division the word "applicare".

Next he introduces the rules

$$B - (C \pm D) = B - C \mp D; \quad B \text{ in } (C \pm D) = B \text{ in } C \pm B \text{ in } D,$$

as well as operations on fractions, written by means of letters, e.g.,

$$\frac{B\text{pl}}{D} + Z = \frac{B\text{pl} + Z \text{ in } D}{D}.$$

Viète's next treatise was *Ad logisticam speciosam notae priores*, which appeared only in 1646 as part of his collected works. In it he set down some of the most important algebraic formulas, such as:

$$(A + B)^n = A^n \pm nA^{n-1}B + \cdots \pm B^n, \quad n = 2, 3, 4, 5;$$
$$A^n + B^n = (A + B)(A^{n-1} - A^{n-2}B + \cdots \pm B^{n-1}), \quad n = 3, 5;$$
$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1}), \quad n = 2, 3, 4, 5.$$

Viète's literal calculus was perfected by René Descartes (1596–1650), who dispensed with the principle of homogeneity and gave the literal calculus its modern form.

2. The contribution of René Descartes (1596–1650). The 16th century was marked by remarkable achievements in algebra and was followed by a period of relative calm in this area. Most of the energy of 17th-century mathematicians was absorbed by infinitesimal analysis, which was created at that time. Nevertheless, while inconspicuous at first sight, profound changes were taking place in algebra that can be characterized by one word—arithmetization.

The first steps in this direction were taken by the famous philosopher and mathematician René Descartes (1596–1650). In his *Geometry* (the fourth part of his 1637 *Discourse on method*), whose essential content was the reduction of geometry to algebra or, in other words, the creation of analytic geometry, he first of all transformed Viète's calculus of magnitudes (logistica speciosa). Descartes represented all magnitudes by segments, and constructed a calculus of segments that differed essentially from the one that was used in antiquity and that formed the basis of Viète's construction. Descartes' idea was that the operations on segments should be a faithful replica of (we would say "should be isomorphic to") the operations on rational numbers. Whereas the ancients and Viète regarded the product of two segment magnitudes as an area, i.e., as a magnitude of dimension 2, Descartes stipulated that it was to be a segment. To this end, he introduced a unit segment—which we will denote by e—and defined the product of segments a and b as the segment c that was the fourth proportional to the segments e, a, and b. Specifically (see Figure 1), he constructed an arbitrary angle ABC, and laid off the

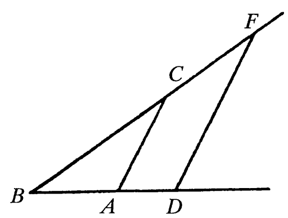

Figure 1

segments $AB = e$, $BD = b$ and $BC = a$. Then he joined A to C, drew $DF \| AC$, and obtained the segment $BF = c = ab$. This meant that the product belonged to the same domain of magnitudes (segments) as the factors. Division was defined analogously: to divide $BF = c$ by $BD = b$ we lay off from the vertex B of the angle the segment $BA = e$, join F to D, and draw AC parallel to DF. The segment BC is the required quotient. In this way, Descartes made the domain of segments into a replica of the semifield $\mathbf{R}^+$. Later he also introduced negative segments (with directions opposite to those of the positive segments) but did not go into the details of operations with negative numbers. Finally, Descartes showed that the operation of extraction of roots (of positive magnitudes) does not take us outside the domain of segments. (We interpolate a comment. Long before Descartes, Bombelli introduced similar rules of operation with segments. Until recently it was thought that he did this in the fourth part of his manuscript published only in the 20th century. However, G. S. Smirnova showed recently that such operations on segments occur also in the parts of Bombelli's *Algebra* published in 1572, i.e., during his lifetime.) To extract the root of $c = BF$, Descartes extended this segment, laid off $FA = e$ on the extension, drew a semicircle with diameter BA, and erected at F the perpendicular to BA. If I is its intersection with the semicircle, then $FI = \sqrt{c}$.

Descartes' calculus was of tremendous significance for the subsequent development of algebra. It not only brought segments closer to numbers but also lent to algebra the simplicity and operativeness that we take advantage of to this day. Another convention introduced by Descartes and used to this day is denoting unknowns by the last letters of the alphabet: x, y, z, and knowns by the first letters: a, b, c. The only difference between Descartes' symbolism and modern symbolism is his equality sign: $\propto$.

Essentially, it was Descartes who established the isomorphism between the domain of segments and the semifield $\mathbf{R}^+$ of real numbers. However, he gave no general definition of number. This was done by Newton in his *Universal Arithmetic* in which the construction of algebra on the basis of arithmetic reached its completion. He wrote: "Computation is conducted either by means of numbers, as in ordinary arithmetic, or through general variables, as is the habit of analytical mathematicians." And further: "Yet arithmetic is so instrumental to algebra in all its operations that they seem jointly to constitute but a single, complete computing science, and for that reason I shall explain both together."

Newton immediately gives a general definition of number. We recall that in antiquity number denoted a collection of units (i.e., natural numbers), and that ratios of numbers (rational numbers) and ratios of like quantities (real numbers) were not regarded as numbers. Claudius Ptolemy (2nd century AD) and Arab mathematicians did identify ratios with numbers, but in 16th- and 17th-century Europe the Euclidean tradition was still very strong. Newton was the first to break with it openly. He wrote:

> By a 'number' we understand not so much a multitude of units as the abstract ratio of any quantity to another quantity which is considered to be unity. It is threefold: integral, fractional, and surd. An integer is measured by unity, a fraction by a submultiple part of unity, while a surd is incommensurable with unity.

With characteristic brevity, Newton goes on to define negative numbers:

> Quantities are either positive, that is, greater than zero, or negative, that is, less than zero. ... in geometry, if a line drawn with advancing motion in some direction be considered as positive, then its negative is one drawn retreating in the opposite direction.
>
> To denote a negative quantity... the sign $-$ is usually prefixed, to a positive one the sign $+$.

Then Newton formulates rules of operation with relative numbers. We quote his multiplication rule: "A product is positive if both factors are positive or both negative and it is negative otherwise."

He provides no "justifications" for these rules.

(*Translator's note*. The preceding quotations are taken from D. T. Whiteside's English translation of Newton's *Arithmetica universalis*.)

Thus Viète's elaborate domain of magnitudes was replaced in the 17th century by the field of real numbers and arithmetic formed the foundation of algebra.

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