PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman,** and **Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Paul Bracken, Duane M. Broline, Ezra A. Brown, Richard T. Bumby, Glenn G. Chappell, Randall Dougherty, Roger B. Eggleton, Ira M. Gessel, Bart Goddard, Jerrold R. Griggs, Douglas A. Hensley, Richard Holzsager, John R. Isbell, Robert Israel, Kiran S. Kedlaya, Murray S. Klamkin, Fred Kochman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Charles Vanden Eynden, and William E. Watkins.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before August 31, 1999; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10718. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.* Let p be a prime number with $p \equiv 7 \pmod 8$, and let $L_p = \{1, 2, 3, \ldots, (p-1)/2\}$. Prove that the sum of the quadratic residues modulo p in L_p equals the sum of the quadratic nonresidues modulo p in L_p. For example, the quadratic residues in L_{23} are 1, 2, 3, 4, 6, 8, and 9, and the quadratic nonresidues in L_{23} are 5, 7, 10, and 11. Both lists sum to 33.

10719. *Proposed by Jean Anglesio, Garches, France.* Let A, I, and G be three points in the plane. Let M denote the point 2/3 of the way from A to I, and let U and V be the circles of radius $|AM|$ each of which is tangent to AI at M. Show that when G is outside both U and V, there are precisely two triangles ABC with incenter I and centroid G. Provide a Euclidean construction for them. Show that when G is in the interior of U or V, there does not exist a triangle ABC with incenter I and centroid G.

10720. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* A "binary maze" is a directed graph in which exactly two arcs lead from each vertex, one labeled 0 and one labeled 1. If $b_1, b_2, \ldots, b_m$ is any sequence of 0s and 1s and v is any vertex, let $vb_1b_2\cdots b_m$ be the vertex reached beginning at v and traversing arcs labeled $b_1, b_2, \ldots, b_m$ in order. A sequence $b_1, b_2, \ldots, b_m$ of 0s and 1s is a *universal exploration sequence* of order n if, for every strongly connected binary maze on n vertices and every vertex v, the sequence

$$v, vb_1, vb_1b_2, \ldots, vb_1b_2\cdots b_m$$

includes every vertex of the maze. For example, 01 is a universal exploration sequence of order 2, and it can be shown that 0110100 is universal of order 3.

(a) Prove that universal exploration sequences of all orders exist.

(b)* Find a good estimate for the asymptotic length of the shortest such sequence of order n.

10721. *Proposed by Daniel A. Sidney, Massachusetts Institute of Technology, Cambridge, MA.* Let $f(x) = \sin x/x$, and let m and n be nonnegative integers. Compute

$$\int_0^\infty \frac{d^m}{dx^m} f(x)\, \frac{d^n}{dx^n} f(x)\, dx.$$