varying the z_i does not change α_1, so we may choose a sequence for which β_1 approaches its infimum. A subsequence either degenerates to a lower dimensional simplex or leads to a counterexample with β_1 minimal. If the limit is degenerate, then a computation shows that there is a counterexample for lower m, contradicting the minimality of m.

Therefore we may consider a counterexample that has β_1 minimal under varying the z_i. Assume that F_1 lies in the affine subspace $S = \{(a, x_2, \ldots, x_m)\}$, and let P_1^* be the projection of P_1 into this subspace. Arguing as for the 3-dimensional case, we see that P_1^* is in the interior of F_1 and is equidistant from all the faces of F_1. Let this common distance be b. Let F be a face of F_1, let T be the $(m-2)$-dimensional affine subspace containing F, and let Q be the orthogonal projection of P_1 into T. Let $f(r)dr$ be the solid angle generated from P_1^* by the points of T whose distance from Q is between r and $r+dr$. Let $S(r)$ be the sphere of radius r about Q in T, and define $g_F(r) = \text{area}(F \cap S(r))/\text{area}(S(r))$. Note that $g_F(r)$ is nonincreasing, by convexity of F. If a solid angle Φ in S with vertex P_1^* meets T at a distance from Q of between r and $r+dr$, then let $h_b(r)$ be the measure of the solid angle from P_1 generated by the portion of Φ bounded by T. With these definitions, F generates a solid angle of $\int_0^\infty g_F(r)f(r)dr$ from P_1^* in S, and the portion of F_1 between F and P_1^* generates a solid angle of $\int_0^\infty g_F(r)h_b(r)f(r)dr$ from P_1.

Since $g_F(r)$ is nonincreasing and nonconstant, and since $h_b(r)$ is increasing, we have

$$\int_0^\infty f(r)dr \int_0^\infty g_F(r)h_b(r)f(r)dr < \int_0^\infty h_b(r)f(r)dr \int_0^\infty g_F(r)f(r)dr.$$

Let A_t be the $(t-1)$-dimensional area of the t-dimensional sphere of radius 1. Summing the last inequality over all faces of F_1 gives

$$A_m\alpha_1 \int_0^\infty f(r)dr < A_{m-1} \int_0^\infty h_b(r)f(r)dr. \tag{2}$$

The same calculation applies if F_1 is replaced by the slab $G = \{(a, x_2, \ldots, x_m) \colon |x_2| \le b\}$, except that (1) we now get equality, since for both faces H of G, the function g_H is identically 1, and (2) α_1 is replaced by ϕ, the probability that the ray from O through a random point $v = (y_1, \ldots, y_m)$ on the unit sphere hits G. Hence

$$A_m\phi \int_0^\infty f(r)dr = A_{m-1} \int_0^\infty h_b(r)f(r)dr. \tag{3}$$

From (2) and (3), we infer that $\alpha_1 < \phi$. The random ray from O hits G if and only if $y_1 > 0$ and $|y_2|/y_1 \le b/a$. This depends only on the direction of (y_1, y_2), which is uniformly distributed. Thus $\alpha_1 < \phi = \alpha_1'$, where α_1' is the value of α_1 for the (2-dimensional) isosceles triangle J with altitude a and base $2b$. Since J has the same value of β_1 as G has, we are reduced to the 2-dimensional case. Since we can approximate G by F_1, $f(\pi\alpha_1)$ is the best possible lower bound for β_1.

A Tricky Convergence

10614 [1997, 767]. *Proposed by Grigore-Raul Tataru, University of Bucharest, Bucharest, Romania.* Fix $p > 1$. Suppose that $a_1, a_2, \ldots$ is a sequence of positive real numbers such that $a_n a_{n+1} a_{n+2}^p + a_{n+2} - a_n = 0$ for all $n \ge 1$. Show that $\{a_n\}$ is convergent.

Solution by the GCHQ Problems Group, Cheltenham, U. K. Since $a_n - a_{n+2} = a_n a_{n+1} a_{n+2}^p$ is positive, the even and odd subsequences are decreasing and therefore convergent, say to x and y respectively. Taking limits gives $yx^{p+1} = 0 = xy^{p+1}$, so at least one of x and y must be 0. Without loss of generality, we may assume $x = 0$. If $y > 0$, then $a_{2n-1} - a_{2n+1} > y^{p+1}a_{2n}$, so the series $\sum a_{2n}$ converges. Let m be large enough that $a_{2m+1} < 2y$ and $a_{2m} < 1$. Let $\epsilon = a_{2m}$. For $n \ge m$, we have $a_{2n} - a_{2n+2} < 2y\epsilon^{p+1}$, so the number of integers n with $\epsilon/2 \le a_{2n} < \epsilon$ is at least $(\epsilon/2)/(2y\epsilon^{p+1}) = 1/(4y\epsilon^p) > 1/(4y\epsilon)$, and the sum of these terms is therefore at least $1/(8y)$. Thus there are infinitely many disjoint blocks of terms, each of which sums to at least $1/(8y)$. This contradicts the fact that $\sum a_{2n}$ converges. Therefore $x = y = 0$, and $a_n \to 0$.

Solved also by J. Anglesio (France), S. S. Kim (Korea), K.-W. Lau (Hong Kong), J. H. Lindsey II, M. Shemesh (Israel), NSA Problems Group, and the proposer.

Tails of an Alternating Series

10624 [1997, 871]. *Proposed by William F. Trench, Trinity University, San Antonio, TX.* Suppose that $a_0 > a_1 > a_2 > \cdots$ and $\lim_{n\to\infty} a_n = 0$. Define $S_n = \sum_{j=n}^{\infty}(-1)^{j-n}a_j = a_n - a_{n+1} + a_{n+2} - \cdots$. Show that $\sum a_n S_n < \infty$ if and only if $\sum a_n^2 < \infty$.

Solution by Douglas B. Tyler, Hughes Aircraft Company, El Segundo, CA. More generally, we prove that the three series $\sum S_n^2$, $\sum a_n S_n$, and $\sum a_n^2$ converge or diverge together. By the alternating series test, S_n exists and satisfies $0 < S_n < a_n$. Thus $\sum S_n^2 < \sum a_n S_n < \sum a_n^2$. So it suffices to show that finiteness of $\sum S_n^2$ implies finiteness of $\sum a_n^2$. To prove it, use $S_n = a_n - S_{n+1}$ and the inequality $(x+y)^2 \le 2(x^2+y^2)$ to infer

$$\sum a_n^2 = \sum (S_n + S_{n+1})^2 \le \sum 2(S_n^2 + S_{n+1}^2) = 2\sum S_n^2 + 2\sum S_{n+1}^2 < 4\sum S_n^2.$$

Solved also by S. A. Ali, K. F. Andersen (Canada), R. Barbara (France), G. L. Brody (U. K.), P. Bracken (Canada), P. Budney, D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. N. Deshpande (India), Z. Franco, T. Goebeler & T. Siemers, T. Hermann, V. Hernandez & J. Martin (Spain), S. S. Kim (Korea), R. A. Kopas, O. Kouba (Syria), M. Kumar (India), J. H. Lindsey II, N. Lord (U. K.), P. Mengert, R. Mortini (France), M. Omarjee (France), L. Opperman, G. Peng, H. Salle (The Netherlands), K. Schilling, H.-J. Seiffert (Germany), N. C. Singer, A. Stenger, S. J. Swiniarski, N. S. Thornber, A. Tissier (France), P. Trojovsky (Czech Republic), M. Woltermann, C. Xiong, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.

Cosecants and Near-Integers

10630 [1997, 975]. *Proposed by Richard Stong, Rice University, Houston, TX.* It is possible to show that $\csc(3\pi/29) - \csc(10\pi/29) = 1.999989433\ldots$. Prove that there are no integers j, k, n with n odd satisfying $\csc(j\pi/n) - \csc(k\pi/n) = 2$.

Solution by Allen Stenger, Tustin, CA. Suppose that there exist such integers j, k, n. Let $\omega_m = e^{2\pi i/m}$ be a primitive mth root of unity. Then

$$\frac{2i}{\omega_{2n}^{j} - \omega_{2n}^{-j}} - \frac{2i}{\omega_{2n}^{k} - \omega_{2n}^{-k}} = 2,$$

which rearranges to

$$i = \frac{(\omega_{2n}^{k} - \omega_{2n}^{-k})(\omega_{2n}^{j} - \omega_{2n}^{-j})}{(\omega_{2n}^{k} - \omega_{2n}^{-k}) - (\omega_{2n}^{j} - \omega_{2n}^{-j})}.$$

This implies that i is in $\mathbb{Q}(\omega_{2n})$, the cyclotomic field of order $2n$ over the rationals.

We now claim that adjoining i to $\mathbb{Q}(\omega_{2n})$ produces $\mathbb{Q}(\omega_{2n}, i) = \mathbb{Q}(\omega_{4n})$. To show this, observe that $\omega_{2n} = \omega_{4n}^2 \in \mathbb{Q}(\omega_{4n})$, $i = \omega_{4n}^n \in \mathbb{Q}(\omega_{4n})$, and $\omega_{4n} = \omega_{4n}^{n+1}/\omega_{4n}^n = \omega_{2n}^{(n+1)/2}/i \in \mathbb{Q}(\omega_{2n}, i)$. (This is where we need the hypothesis that n is odd.)

This produces the contradiction: If it were true that $i \in \mathbb{Q}(\omega_{2n})$, then we would have $\mathbb{Q}(\omega_{2n}) = \mathbb{Q}(\omega_{2n}, i) = \mathbb{Q}(\omega_{4n})$, but this is impossible because the degrees of $\mathbb{Q}(\omega_{2n})$ and $\mathbb{Q}(\omega_{4n})$ over $\mathbb{Q}$ are known to be $\phi(2n)$ and $\phi(4n) = 2\phi(2n)$, respectively.

Editorial comment. Gerry Myerson proved more: If j, k, n are positive integers with no common factor and $\csc(j\pi/n) - \csc(k\pi/n)$ is a nonzero rational number r, then n is 2 or 6 and r is ± 1, ± 2, ± 3, or ± 4.

Solved also by R. J. Chapman (U. K.), J. H. Lindsey II, G. Myerson (Australia), and the proposer.