the sum of these terms is therefore at least $1/(8y)$. Thus there are infinitely many disjoint blocks of terms, each of which sums to at least $1/(8y)$. This contradicts the fact that $\sum a_{2n}$ converges. Therefore $x = y = 0$, and $a_n \to 0$.

Solved also by J. Anglesio (France), S. S. Kim (Korea), K.-W. Lau (Hong Kong), J. H. Lindsey II, M. Shemesh (Israel), NSA Problems Group, and the proposer.

Tails of an Alternating Series

10624 [1997, 871]. *Proposed by William F. Trench, Trinity University, San Antonio, TX.* Suppose that $a_0 > a_1 > a_2 > \cdots$ and $\lim_{n\to\infty} a_n = 0$. Define $S_n = \sum_{j=n}^{\infty}(-1)^{j-n}a_j = a_n - a_{n+1} + a_{n+2} - \cdots$. Show that $\sum a_n S_n < \infty$ if and only if $\sum a_n^2 < \infty$.

Solution by Douglas B. Tyler, Hughes Aircraft Company, El Segundo, CA. More generally, we prove that the three series $\sum S_n^2$, $\sum a_n S_n$, and $\sum a_n^2$ converge or diverge together. By the alternating series test, S_n exists and satisfies $0 < S_n < a_n$. Thus $\sum S_n^2 < \sum a_n S_n < \sum a_n^2$. So it suffices to show that finiteness of $\sum S_n^2$ implies finiteness of $\sum a_n^2$. To prove it, use $S_n = a_n - S_{n+1}$ and the inequality $(x+y)^2 \le 2(x^2+y^2)$ to infer

$$\sum a_n^2 = \sum \left(S_n + S_{n+1}\right)^2 \le \sum 2(S_n^2 + S_{n+1}^2) = 2\sum S_n^2 + 2\sum S_{n+1}^2 < 4\sum S_n^2.$$

Solved also by S. A. Ali, K. F. Andersen (Canada), R. Barbara (France), G. L. Brody (U. K.), P. Bracken (Canada), P. Budney, D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. N. Deshpande (India), Z. Franco, T. Goebeler & T. Siemers, T. Hermann, V. Hernandez & J. Martin (Spain), S. S. Kim (Korea), R. A. Kopas, O. Kouba (Syria), M. Kumar (India), J. H. Lindsey II, N. Lord (U. K.), P. Mengert, R. Mortini (France), M. Omarjee (France), L. Opperman, G. Peng, H. Salle (The Netherlands), K. Schilling, H.-J. Seiffert (Germany), N. C. Singer, A. Stenger, S. J. Swiniarski, N. S. Thornber, A. Tissier (France), P. Trojovsky (Czech Republic), M. Woltermann, C. Xiong, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.

Cosecants and Near-Integers

10630 [1997, 975]. *Proposed by Richard Stong, Rice University, Houston, TX.* It is possible to show that $\csc(3\pi/29) - \csc(10\pi/29) = 1.999989433\ldots$. Prove that there are no integers j, k, n with n odd satisfying $\csc(j\pi/n) - \csc(k\pi/n) = 2$.

Solution by Allen Stenger, Tustin, CA. Suppose that there exist such integers j, k, n. Let $\omega_m = e^{2\pi i/m}$ be a primitive mth root of unity. Then

$$\frac{2i}{\omega_{2n}^{j} - \omega_{2n}^{-j}} - \frac{2i}{\omega_{2n}^{k} - \omega_{2n}^{-k}} = 2,$$

which rearranges to

$$i = \frac{(\omega_{2n}^{k} - \omega_{2n}^{-k})(\omega_{2n}^{j} - \omega_{2n}^{-j})}{(\omega_{2n}^{k} - \omega_{2n}^{-k}) - (\omega_{2n}^{j} - \omega_{2n}^{-j})}.$$

This implies that i is in $\mathbb{Q}(\omega_{2n})$, the cyclotomic field of order $2n$ over the rationals.

We now claim that adjoining i to $\mathbb{Q}(\omega_{2n})$ produces $\mathbb{Q}(\omega_{2n}, i) = \mathbb{Q}(\omega_{4n})$. To show this, observe that $\omega_{2n} = \omega_{4n}^2 \in \mathbb{Q}(\omega_{4n})$, $i = \omega_{4n}^n \in \mathbb{Q}(\omega_{4n})$, and $\omega_{4n} = \omega_{4n}^{n+1}/\omega_{4n}^n = \omega_{2n}^{(n+1)/2}/i \in \mathbb{Q}(\omega_{2n}, i)$. (This is where we need the hypothesis that n is odd.)

This produces the contradiction: If it were true that $i \in \mathbb{Q}(\omega_{2n})$, then we would have $\mathbb{Q}(\omega_{2n}) = \mathbb{Q}(\omega_{2n}, i) = \mathbb{Q}(\omega_{4n})$, but this is impossible because the degrees of $\mathbb{Q}(\omega_{2n})$ and $\mathbb{Q}(\omega_{4n})$ over $\mathbb{Q}$ are known to be $\phi(2n)$ and $\phi(4n) = 2\phi(2n)$, respectively.

Editorial comment. Gerry Myerson proved more: If j, k, n are positive integers with no common factor and $\csc(j\pi/n) - \csc(k\pi/n)$ is a nonzero rational number r, then n is 2 or 6 and r is ±1, ±2, ±3, or ±4.

Solved also by R. J. Chapman (U. K.), J. H. Lindsey II, G. Myerson (Australia), and the proposer.