NOTES

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An Inequality Relating the Circumradius and Diameter of Two-Dimensional Lattice-Point-Free Convex Bodies

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1. Introduction. The first ideas of convex sets date as far back as Archimedes but it was not until the end of the last century that a systematic study gave rise to the subject as an independent branch in mathematics. In particular, many geometric inequalities for convex bodies have been obtained; see **[1]**, **[8]**, **[10]**, **[11]**, and **[12]**.

At the turn of the century, Minkowski **[6]** published his famous Convex Body Theorem, which is the basis for the geometry of numbers. The idea is to interpret integer solutions of equations or inequalities as points with integer coordinates (lattice points). Minkowski's work provides the link between the theory of convex sets and the geometry of numbers. Minkowski's Theorem states that if a convex set in the plane is symmetric about the origin and its interior contains no other lattice point, then its area is at most 4. By studying other geometrical functionals defined on a convex set and varying the conditions on Minkowski's Theorem, many inequalities may be obtained for lattice constrained sets; see **[2]**, **[3]**, **[4]**, **[5]**, and **[9]**.

In this note, we prove an inequality concerning the circumradius and diameter of a planar convex set. We use this inequality to obtain a corresponding result for a lattice-point-free convex set.

2. Notation and Definitions. Throughout this note, K denotes a compact, convex set in the plane. The *circumradius* of K, denoted by $R(K) = R$, is the radius of the smallest disk containing K. The *inradius* of K, denoted by $r(K) = r$, is the radius of the largest disk contained in K. The diameter of K, denoted by $D(K) = D$, is the maximal distance between any two points of K. The *width of K taken in a particular direction* is the distance between the two parallel tangents to K perpendicular to the given direction. The *width* of K denoted by $w(K) = w$, is the minimum of widths taken in all directions.

3. Motivation. The result in this note is motivated by an inequality by Blaschke, which states $w \leq 3r$ for any planar convex set K, with equality when and only when K is an equilateral triangle **[11**, p. 18]. This inequality may be rewritten as

$$w - 2r \leq \frac{w}{3}. \tag{1}$$

If K contains no interior lattice point, we have the following result by Scott **[7]**:

$$w \leq \frac{1}{2}(2 + \sqrt{3}), \tag{2}$$

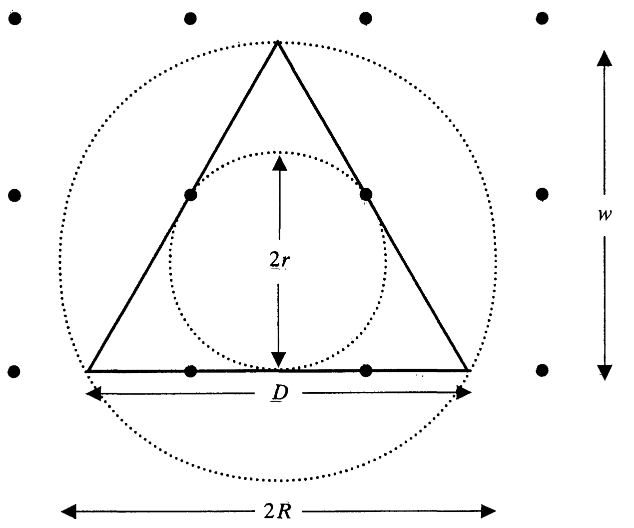

Figure 1. Equilateral triangle with no interior lattice points

with equality when and only when K is congruent to the equilateral triangle shown in Figure 1.

Combining (2) with (1), we have

$$w - 2r \leq \frac{1}{6}(2 + \sqrt{3}), \tag{3}$$

with equality when and only when K is congruent to the equilateral triangle shown in Figure 1.

In this note, we prove 'duals' of (1) and (3):

Theorem. *Let K be a planar, compact, convex set. Then*

$$2R - D \leq \frac{2}{3}(2 - \sqrt{3})w, \tag{4}$$

with equality when and only when K is an equilateral triangle. If K contains no lattice point in its interior, then

$$2R - D \leq \frac{1}{3}, \tag{5}$$

with equality when and only when K is congruent to the equilateral triangle shown in Figure 1.

4. Proof of the Theorem. We may assume that the interior of K is nonempty, otherwise, either $K = \varnothing$ or K is a line segment. If $K = \varnothing$, then (4) is trivially true. If K is a line segment then $D = 2R$, $w = 0$, and again, (4) is trivially true. Hence we may assume that $r \neq 0$. It follows that $w \neq 0$. We now define and seek to maximize the functional

$$f(K) = \frac{1}{w(K)}(2R(K) - D(K)) = \frac{1}{w}(2R - D).$$

Clearly, $f(K) \geq 0$ since $D \leq 2R$. We first recall that the circumcircle of a set K either contains two diametrically opposite points of K or else it contains three points on the boundary of K that form the vertices of an acute-angled triangle [**11**, p. 59]. In the first case, $2R = D$ and $f(K) = 0$, so K is not maximal. Hence we may assume that K contains an acute-angled triangle T with $R(T) = R(K)$. Furthermore, since T is contained in K, $D(T) \leq D(K)$ and $w(T) \leq w(K)$. It follows that $f(K) \leq f(T)$. Hence it suffices to maximize $f(K)$ for acute-angled triangles T.

Let $T = \Delta XYZ$ be an acute-angled triangle with $\angle Y \leq \angle X \leq \angle Z$, as shown in Figure 2. Since $\angle Z$ is the largest angle, it follows that $XY = D$. We first apply to T

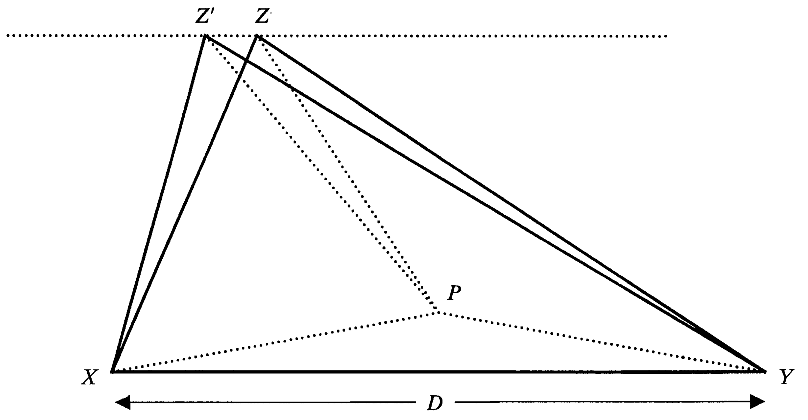

Figure 2. Shear applied to the triangle T

a shear parallel to XY to obtain the triangle $T' = XYZ'$ with $YZ' = XY = D$. Let P and P' be the circumcentres of T and T' respectively. Since P and P' both lie on the perpendicular bisector of the line segment XY, and since $PZ' > PZ = R(T)$, it follows that P' is farther away from XY than the point P. Hence $R(T') > R(T)$. Furthermore $D(T') = D(T)$ and $w(T') = w(T)$. It follows that $f(T') \geq f(T)$. Hence we need consider only those cases for which T is an isosceles triangle with vertex angle at Y. In this case $\angle X = \angle Z = \alpha \geq \pi/3$.

We note that $w = D \sin 2\alpha$ and the sine rule gives $2R = D/\sin\alpha$. Hence we have

$$f(k) = \frac{1}{w}\left(\frac{1}{\sin\alpha} - 1\right)D = \left(\frac{1}{\sin\alpha} - 1\right)\left(\frac{1}{\sin 2\alpha}\right).$$

Letting $t = \tan\alpha$ gives

$$f(K) = \left(\frac{\sqrt{1+t^2}}{t} - 1\right)\left(\frac{1+t^2}{2t}\right) = \frac{1}{2}\left(\sqrt{1+t^2} - t\right)\left(\frac{1+t^2}{t^2}\right)$$
$$= \frac{1}{2}\left(\sqrt{1+t^2} - t\right)\left(\frac{1}{t^2} + 1\right) = \frac{1}{2}g(t)h(t).$$

We note that

$$g(t) = \sqrt{1+t^2} - t > 0, \qquad g'(t) = \frac{1}{\sqrt{1+t^2}} - 1 < 0,$$
$$h(t) = \frac{1}{t^2} + 1 > 0, \qquad h'(t) = -\frac{2}{t^3} < 0.$$

Since $f(K)$ is a product of positive, decreasing functions of t, it is itself a positive, decreasing function of t. Since $\alpha \geq \pi/3$, we have $t \geq \sqrt{3}$. Hence the maximal value of $f(K)$ is attained when $t = \sqrt{3}$, that is, when T is an equilateral triangle. In this case

$$f(K) = \frac{1}{w}(2R - D) \leq \frac{2}{3}(2 - \sqrt{3}).$$

Now suppose that K has no lattice point in its interior. Combining (4) with (2) gives

$$2R - D \leq \frac{1}{3},$$

with equality when and only when K is congruent to the equilateral triangle shown in Figure 1. ∎

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Cutting a Polyomino into Triangles of Equal Areas

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In 1970 Monsky proved that a square cannot be cut into an odd number of triangles of equal areas [**1**], [**6**, p. 118]. This result has been generalized four times. Mead proved that when an n-dimensional cube is cut into simplices of equal volumes, the number of simplices is a multiple of $n!$ [**2**]. Kasimatis proved that when a regular n-gon, $n \geq 5$, is cut into triangles of equal areas, the number of triangles is a multiple of n [**3**]. Stein proved that the theorem about the square