A Short Proof of Turan's Theorem

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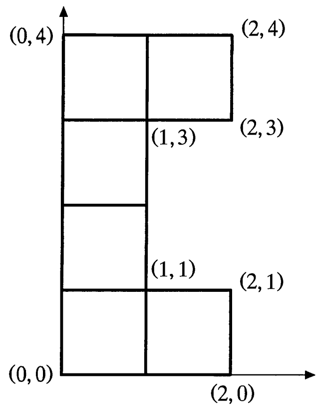

Figure 1

be that the number of triangles would be a multiple of 4. It is necessary to transform the polyomino so that the image has an area A for which $\varphi(A) \leq 0$. Applying the transformation $(x, y) \to (x, y/2)$ followed by the translation by $(1, 1)$ produces a labeling described in the hypothesis of the lemma, as may be verified.

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Extremal graph theory is the search for the thresholds in edge density where substructures of interest are forced to appear in graphs. The canonical extremal theorem involving structure S is of the type: If G is a graph with n vertices containing no S, then G has no more than $f(n)$ edges. The genesis moment of extremal graph theory occurred in 1941 with Turán's article **[1]** in which he proved the canonical extremal result for $S = K_r$, a complete graph with r vertices. The purpose of this note is to provide a new and perhaps shorter proof than has previously been noticed.

Theorem (Turán, 1941). *Graphs with n vertices containing no K_r have no more than $(r-2)n^2/(2r-2)$ edges , for $r \geq 2$.*

Proof: Induct on r. If $r = 2$, the result is obvious. Now if the statement is true for K_r-free graphs it must be shown that K_{r+1}-free graphs have no more than $(r-1)n^2/2r$ edges. Let G be such a graph, and let x be the number of vertices in a largest K_r-free induced subgraph of G. Since the neighbors of any vertex induce a K_r-free subgraph, no vertex of G has degree exceeding x. Let A be a largest induced K_r-free subgraph of G. By induction, there are at most $(r-2)x^2/(2r-2)$ edges in A. Each edge of G not in A is incident with at least one of the $n - x$ vertices not in A, so summing the degrees of these vertices counts each such edge at least once. Hence there are at most $x(n-x)$ such edges and so G has at most $(r-2)x^2/(2r-2) + x(n-x)$ edges. Since

$$\frac{r-2}{2r-2}(x^2) + x(n-x) = \frac{r-1}{2r}n^2 - \frac{r}{2r-2}\left(x - \frac{(r-1)n}{r}\right)^2,$$

the result follows. ∎

Turán's theorem continues, in every graph theory textbook, to be the centerpiece of the presentation of extremal graph theory. For this reason, we hope our short proof will be found worthwhile.

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A Characterization of the Set of Points of Continuity of a Real Function

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In this note, we prove the converse of the following well known result: the set of points of continuity of an arbitrary real valued function on a metric space is a countable intersection of open sets [**1**, p. 58].

Lemma. *If X is a nonempty metric space without isolated points, then X has a dense subset A whose complement is also dense in X.*

Proof: Call a set $S \subset X$ an ϵ-net if (a) $d(x, y) \geq \epsilon$ for any two distinct points x, y of S, and (b) S is maximal with respect to (a). Zorn's Lemma yields that ϵ-nets exist for every $\epsilon > 0$. Suppose we have disjoint sets $S_1, S_2, \ldots, S_k$, where each S_i is an $(1/i)$-net. The complement of $S_1 \cup \cdots \cup S_k$ is then nonempty and has no isolated points, and therefore there is an S_{k+1}, disjoint from $S_1 \cup \cdots \cup S_k$, which is an $(1/(k+1))$-net. Then $A = \bigcup_{n=1}^{\infty} S_{2n}$ and $B = \bigcup_{n=1}^{\infty} S_{2n-1}$ are disjoint, and both are dense in X.

Theorem. *Let X be a nonempty metric space without isolated points. If G is a countable intersection of open sets, then there is a function $\phi(x)$ which is continuous exactly on G.*