



A Short Proof of Turan's Theorem

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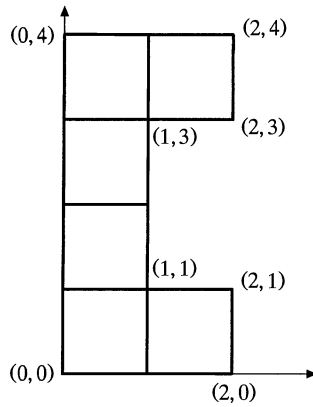


Figure 1

be that the number of triangles would be a multiple of 4. It is necessary to transform the polyomino so that the image has an area A for which $\varphi(A) \leq 0$. Applying the transformation $(x, y) \rightarrow (x, y/2)$ followed by the translation by $(1, 1)$ produces a labeling described in the hypothesis of the lemma, as may be verified.

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A Short Proof of Turán’s Theorem

William Staton

Extremal graph theory is the search for the thresholds in edge density where substructures of interest are forced to appear in graphs. The canonical extremal theorem involving structure S is of the type: If G is a graph with n vertices containing no S , then G has no more than $f(n)$ edges. The genesis moment of extremal graph theory occurred in 1941 with Turán’s article [1] in which he proved the canonical extremal result for $S = K_r$, a complete graph with r vertices. The purpose of this note is to provide a new and perhaps shorter proof than has previously been noticed.

Theorem (Turán, 1941). *Graphs with n vertices containing no K_r have no more than $(r - 2)n^2 / (2r - 2)$ edges, for $r \geq 2$.*

Proof: Induct on r . If $r = 2$, the result is obvious. Now if the statement is true for K_r -free graphs it must be shown that K_{r+1} -free graphs have no more than $(r - 1)n^2/2r$ edges. Let G be such a graph, and let x be the number of vertices in a largest K_r -free induced subgraph of G . Since the neighbors of any vertex induce a K_r -free subgraph, no vertex of G has degree exceeding x . Let A be a largest induced K_r -free subgraph of G . By induction, there are at most $(r - 2)x^2/(2r - 2)$ edges in A . Each edge of G not in A is incident with at least one of the $n - x$ vertices not in A , so summing the degrees of these vertices counts each such edge at least once. Hence there are at most $x(n - x)$ such edges and so G has at most $(r - 2)x^2/(2r - 2) + x(n - x)$ edges. Since

$$\frac{r - 2}{2r - 2}(x^2) + x(n - x) = \frac{r - 1}{2r}n^2 - \frac{r}{2r - 2}\left(x - \frac{(r - 1)n}{r}\right)^2,$$

the result follows. ■

Turán's theorem continues, in every graph theory textbook, to be the centerpiece of the presentation of extremal graph theory. For this reason, we hope our short proof will be found worthwhile.

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A Characterization of the Set of Points of Continuity of a Real Function

Sung Soo Kim

In this note, we prove the converse of the following well known result: the set of points of continuity of an arbitrary real valued function on a metric space is a countable intersection of open sets [1, p. 58].

Lemma. *If X is a nonempty metric space without isolated points, then X has a dense subset A whose complement is also dense in X .*

Proof: Call a set $S \subset X$ an ϵ -net if (a) $d(x, y) \geq \epsilon$ for any two distinct points x, y of S , and (b) S is maximal with respect to (a). Zorn's Lemma yields that ϵ -nets exist for every $\epsilon > 0$. Suppose we have disjoint sets S_1, S_2, \dots, S_k , where each S_i is an $(1/i)$ -net. The complement of $S_1 \cup \dots \cup S_k$ is then nonempty and has no isolated points, and therefore there is an S_{k+1} , disjoint from $S_1 \cup \dots \cup S_k$, which is an $(1/(k + 1))$ -net. Then $A = \bigcup_{n=1}^{\infty} S_{2n}$ and $B = \bigcup_{n=1}^{\infty} S_{2n-1}$ are disjoint, and both are dense in X .

Theorem. *Let X be a nonempty metric space without isolated points. If G is a countable intersection of open sets, then there is a function $\phi(x)$ which is continuous exactly on G .*