10730. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Fix an integer $n \geq 2$. Determine the largest constant $C(n)$ such that

$$\sum_{1 \leq i < j \leq n} \left(x_j - x_i\right)^2 \geq C(n) \cdot \min_{1 \leq i < n} \left(x_{i+1} - x_i\right)^2$$

for all real numbers $x_1 < x_2 < \cdots < x_n$.

10731. *Proposed by M. J. Pelling, London, England.* Let A be an n-by-n real symmetric matrix, and consider the quadratic form $Q(x) = x^T Ax$ for $x \in \mathbb{R}^n$. Let C be the cube $[-1, 1]^n$. Prove that $\max_{x \in C} Q(x)$ is at least as large as the sum of the positive real eigenvalues of A.

SOLUTIONS

Connected Sets of Periodic Functions

10434 [1995, 170]. *Proposed by Daniel Goffinet, Saint Étienne, France.* Let P be the set of nonconstant periodic mappings from $\mathbb{R}$ to $\mathbb{R}$, endowed with the topology derived from the supremum norm. Find the components of P.

Composite solution I by Kiran S. Kedlaya, Massachusetts Institute of Technology, Cambridge, MA, Kenneth Schilling, University of Michigan, Flint, MI, and Arlo W. Schurle, University of Guam, Mangilao, Guam. For any function $f\colon \mathbb{R} \to \mathbb{R}$, define $\|f\|$ to be $\sup\{\, |f(x)| : x \in \mathbb{R}\}$, which is taken to be ∞ when the set of values of f is unbounded.

We first show that f and g are in different components of P if $\|f - g\| = \infty$. Let $B_g = \{\, k \in P\colon \|k - g\| < \infty\}$. By the triangle inequality B_g is an open set, and if $h \notin B_g$, then the triangle inequality again shows that $\{\, z\colon \|z - h\| < 1\} \cap B_g = \emptyset$. Consequently B_g is both open and closed, and so the component of P containing any given $g \in P$ must lie in B_g.

Conversely, if $f - g$ is bounded for $f, g \in P$, then there is an arc in P joining f to g. First, suppose that f and g have a common period p. The standard path $k_t(x) = (1 - t)f(x) + tg(x)$ for $0 \leq t \leq 1$ consists of functions having p as a period, and since $\|f - g\|$ is finite, k_t depends continuously on t. There is a danger that some $k_t(x)$ is a constant function, but this can happen only if f is an affine function of g, that is, there are constants A and B with $f = Ag + B$. In this case, the function $h(x)$ that is equal to $f(x)$ except at integer multiples of p, where it is $f(x) + 1$, is at bounded distance from both f and g and is not an affine function of either. A path from f to g can be obtained by taking the standard path from f to h followed by the standard path from h to g.

Suppose now that f and g have no common period. Let r be a period of f and let s be a period of g. We wish to construct h that has both r and s as periods such that $\|f - h\|$ (and hence also $\|g - h\|$) is finite. To do this, pick an arbitrary set of coset representatives for $\mathbb{R}/(r\mathbb{Z} + s\mathbb{Z})$, define h to agree with f at these values, and extend by periodicity. Then for any x, let $x = y + rm + sn$, where y represents the coset containing x. Then

$$\begin{aligned} |h(x) - f(x)| &= |f(y) - f(y + sn)| \\ &= |f(y) - g(y) + g(y + sn) - f(y + sn)| \leq 2\,\|f - g\| \end{aligned}$$

Since f and h have common period r and $\|f - h\|$ is finite, there is a path from f to h, and since h and g have common period s and $\|h - g\|$ is finite, there is a path from h to g.

Composite solution II by Fredric D. Ancel, University of Wisconsin, Milwaukee, WI, Phil Bowers and John Bryant, The Florida State University, Tallahassee, FL, and the proposer. We assume that "mapping" means "continuous function". Then two functions in P belong to the same component if and only if they have commensurate periods. As in solution I, the components are path-components.

Given $f \in P$ with period $2\pi n$, we form a path from $f(x)$ to $\sin x$ via the homotopy $h_t(x) = (1-t)f(x)+t\sin x$ for $0 \le t \le 1$. Since $\|h_s - h_t\| \le \big(\|f\| + 1\big)|s-t|$, the map $t \mapsto h_t$ is continuous from [0, 1] to the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the topology derived from the supremum norm. Each h_t has $2n\pi$ as a period. This gives the desired path in P if no $h_t(x)$ is constant, i.e., unless $f(x) = A\sin x + B$ with $A < 0$. For such $f(x)$, the path $k_t(x) = (1-t)f(x) - t\sin x$ connects $f(x)$ to $-\sin x = \sin(x+\pi)$. The path $n_t(x) = \sin(x + (1-t)\pi)$ then connects $-\sin x$ to $\sin x$ in P. The continuity of $t \mapsto n_t$ follows from the mean value theorem. If f and g have commensurate periods, say both are a multiple of p, then there are continuous paths in P connecting $f(x)$ to $\sin(2\pi x/p)$ and $g(x)$ to $\sin(2\pi x/p)$, hence there is a path from f to g.

For a fixed real number p, let C_p denote the set of functions in P whose period is a rational multiple of p. We now show that C_p is an open subset of P. Choose $f \in C_p$. Since f is not constant, there are real numbers x and y such that $f(x) < f(y)$. Set $\epsilon = \big(f(y) - f(x)\big)/3$. We show that $g \in P$ and $\|f - g\| < \epsilon$ implies $g \in C_p$. If not, then there is g with $\|f - g\| < \epsilon$ such that the period of g, say q, is an irrational multiple of p. Since f is continuous, there is δ such that f takes the interval $(y-\delta, y+\delta)$ into the interval $(f(y)-\epsilon, f(y)+\epsilon)$. Since $f \in C_p$, f also takes the interval $(y - mp - \delta, y - mp + \delta)$ into $(f(y)-\epsilon, f(y)+\epsilon)$ for each integer m. Since $\|f - g\| < \epsilon$, g takes each $(y - mp - \delta, y - mp + \delta)$ into $(f(y) - 2\epsilon, f(y) + 2\epsilon)$. Since p/q is irrational, the numbers $mp + nq$ for integers m and n are dense in $\mathbb{R}$, so there are integers m and n such that $mp+nq \in (y - x - \delta, y - x + \delta)$, which gives $x + nq \in (y - mp - \delta, y - mp + \delta)$. Thus, $g(x+nq) < f(x) + \epsilon = f(y) - 2\epsilon < g(x+nq)$, a contradiction. This completes the proof that C_p is an open subset of P.

Since the sets C_p partition P into connected open sets, each set C_p is a component of P.

The Plane Covered by Disks

10440 [1995, 273]. *Proposed by Marius Cavachi, Constanta, Romania.* Show that the Euclidean plane cannot be covered with circular disks having mutually disjoint interiors.

Solution I by Sam Northshield, SUNY, Plattsburgh, NY. We show that $\mathbb{R}^k$ ($k \ge 2$) cannot be covered by metric balls having mutually disjoint interiors.

Note that every set of balls with disjoint interiors is countable, since each contains a different point with rational coordinates. Let $\{B_n\colon n \in \mathbb{N}\}$ be a set of closed metric balls in $\mathbb{R}^k$ ($k \ge 2$) with mutually disjoint interiors. A point of intersection of two of the B_n is called an *intersection point.* Since the intersection of two distinct B_n has at most one point, there are only countably many intersection points. Hence we may choose a straight line segment γ with its endpoints in the interiors of two distinct balls and such that γ avoids all intersection points (here is where we need $k \ge 2$). Let C be the set of points of γ that are *not* in the interior of any B_n. Then C is closed and nonempty. Furthermore, any neighborhood of point $x \in C$ must contain another point of C; otherwise x would be an intersection point. Hence C is perfect, and thus uncountable. Now, for any n, the segment γ intersects ∂B_n in at most two points, so there is $x \in C$ not in any B_n. It follows that $\bigcup B_n \neq \mathbb{R}^k$.

Solution II by Simeon T. Stefanov, Sofia, Bulgaria. Suppose the contrary. As in Solution I, there are only countably many intersection points. Let L be a line that avoids these points, and consider the family $\mathcal{F} = \{L \cap B_n\colon n \in \mathbb{N}\}$, a countable cover of L with disjoint closed bounded intervals. To see that no such cover is possible, construct a nested family of compact intervals $\Delta_n \subseteq L$ such that $\Delta_n \cap B_n = \emptyset$, but Δ_n meets at least two intervals in $\mathcal{F}$. Then $\bigcap \Delta_n$ is nonempty, but no point of this intersection belongs to any set in $\mathcal{F}$.

Editorial comment. Victor Klee noted that the proof that there are only countably many intersection points requires only that the B_n be *rotund* (i.e., strictly convex). His solution followed Solution II, with the last part traced back to W. Sierpiński, Un théorème sur les