Given $f \in P$ with period $2\pi n$, we form a path from $f(x)$ to $\sin x$ via the homotopy $h_t(x) = (1-t)f(x)+t\sin x$ for $0 \le t \le 1$. Since $\|h_s - h_t\| \le \big(\|f\|+1\big)|s-t|$, the map $t \mapsto h_t$ is continuous from [0, 1] to the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the topology derived from the supremum norm. Each h_t has $2n\pi$ as a period. This gives the desired path in P if no $h_t(x)$ is constant, i.e., unless $f(x) = A\sin x + B$ with $A < 0$. For such $f(x)$, the path $k_t(x) = (1-t)f(x) - t\sin x$ connects $f(x)$ to $-\sin x = \sin(x+\pi)$. The path $n_t(x) = \sin(x+(1-t)\pi)$ then connects $-\sin x$ to $\sin x$ in P. The continuity of $t \mapsto n_t$ follows from the mean value theorem. If f and g have commensurate periods, say both are a multiple of p, then there are continuous paths in P connecting $f(x)$ to $\sin(2\pi x/p)$ and $g(x)$ to $\sin(2\pi x/p)$, hence there is a path from f to g.

For a fixed real number p, let C_p denote the set of functions in P whose period is a rational multiple of p. We now show that C_p is an open subset of P. Choose $f \in C_p$. Since f is not constant, there are real numbers x and y such that $f(x) < f(y)$. Set $\epsilon = \big(f(y) - f(x)\big)/3$. We show that $g \in P$ and $\|f - g\| < \epsilon$ implies $g \in C_p$. If not, then there is g with $\|f - g\| < \epsilon$ such that the period of g, say q, is an irrational multiple of p. Since f is continuous, there is δ such that f takes the interval $(y-\delta, y+\delta)$ into the interval $(f(y)-\epsilon, f(y)+\epsilon)$. Since $f \in C_p$, f also takes the interval $(y-mp-\delta, y-mp+\delta)$ into $(f(y)-\epsilon, f(y)+\epsilon)$ for each integer m. Since $\|f-g\| < \epsilon$, g takes each $(y-mp-\delta, y-mp+\delta)$ into $(f(y)-2\epsilon, f(y)+2\epsilon)$. Since p/q is irrational, the numbers $mp+nq$ for integers m and n are dense in $\mathbb{R}$, so there are integers m and n such that $mp+nq \in (y-x-\delta, y-x+\delta)$, which gives $x+nq \in (y-mp-\delta, y-mp+\delta)$. Thus, $g(x+nq) < f(x)+\epsilon = f(y)-2\epsilon < g(x+nq)$, a contradiction. This completes the proof that C_p is an open subset of P.

Since the sets C_p partition P into connected open sets, each set C_p is a component of P.

The Plane Covered by Disks

10440 [1995, 273]. *Proposed by Marius Cavachi, Constanta, Romania.* Show that the Euclidean plane cannot be covered with circular disks having mutually disjoint interiors.

Solution I by Sam Northshield, SUNY, Plattsburgh, NY. We show that $\mathbb{R}^k$ $(k \ge 2)$ cannot be covered by metric balls having mutually disjoint interiors.

Note that every set of balls with disjoint interiors is countable, since each contains a different point with rational coordinates. Let $\{B_n\colon n \in \mathbb{N}\}$ be a set of closed metric balls in $\mathbb{R}^k$ $(k \ge 2)$ with mutually disjoint interiors. A point of intersection of two of the B_n is called an *intersection point.* Since the intersection of two distinct B_n has at most one point, there are only countably many intersection points. Hence we may choose a straight line segment γ with its endpoints in the interiors of two distinct balls and such that γ avoids all intersection points (here is where we need $k \ge 2$). Let C be the set of points of γ that are *not* in the interior of any B_n. Then C is closed and nonempty. Furthermore, any neighborhood of point $x \in C$ must contain another point of C; otherwise x would be an intersection point. Hence C is perfect, and thus uncountable. Now, for any n, the segment γ intersects ∂B_n in at most two points, so there is $x \in C$ not in any B_n. It follows that $\bigcup B_n \ne \mathbb{R}^k$.

Solution II by Simeon T. Stefanov, Sofia, Bulgaria. Suppose the contrary. As in Solution I, there are only countably many intersection points. Let L be a line that avoids these points, and consider the family $\mathcal{F} = \{L \cap B_n\colon n \in \mathbb{N}\}$, a countable cover of L with disjoint closed bounded intervals. To see that no such cover is possible, construct a nested family of compact intervals $\Delta_n \subseteq L$ such that $\Delta_n \cap B_n = \emptyset$, but Δ_n meets at least two intervals in $\mathcal{F}$. Then $\bigcap \Delta_n$ is nonempty, but no point of this intersection belongs to any set in $\mathcal{F}$.

Editorial comment. Victor Klee noted that the proof that there are only countably many intersection points requires only that the B_n be *rotund* (i.e., strictly convex). His solution followed Solution II, with the last part traced back to W. Sierpiński, Un théorème sur les continus, *Tôhoku Mat. J.* 13 (1918) 300–303. Klee also noted that circular disks are *smooth* (i.e., possess a continuously differentiable parameterization) as well as rotund. For more on smooth tilings, see V. Klee, E. Maluta, and C. Zanco, Tiling with smooth and rotund tiles, *Fund. Math.* 126 (1986) 269–290; V. Klee and C. Tricot, Locally countable plump tilings are flat, *Math. Ann.* 277 (1987) 315–325; and P. M. Gruber, How well can space be packed with smooth bodies? Measure theoretic results, *J. London Math. Soc.* (2) 52 (1995) 1–14.

D. G. Larman, A note on the Besicovich dimension of the closest packing of sphere in R_n, *Proc. Cambridge Philos. Soc.* 62 (1966) 193–195 shows that, in the case of packing of circular disks in the plane, the uncovered set has Hausdorff dimension at least 1.03.

Solved also by G. E. Bredon, P. Budney, J. D. Clemens, J. Cobb, R. Holzsager, A. A. Jagers (The Netherlands), V. Klee, J. H. Lindsey II, O. P. Lossers (The Netherlands), R. Martin (Germany), L. E. Mattics, M. Misiurewicz, I. Namioka, O. Nanyes, C. G. Petalas & T. P. Vidalis (Greece), C. Popescu (Belgium), A. W. Schurle, J. H. Shapiro & T. L. McCoy, A. A. Tarabay & R. Barbara (Lebanon), and the Anchorage Math Solutions Group.

Random Perfect Matchings

10587 [1997, 361]. *Proposed by Joaquín Gómez Rey, Madrid, Spain.* Let K_{2n} be the complete graph on $2n$ vertices. Let P_n be the probability that two random perfect matchings of K_{2n} are disjoint. What is $\lim_{n\to\infty} P_n$?

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela. The limit is $e^{-1/2} \approx 0.60653$. The number of perfect matchings of K_{2n} is $M_n = (2n)!/(2^n n!)$. Given a perfect matching G of K_{2n} and a set J of j edges of G, there are M_{n-j} perfect matchings of K_{2n} containing J. Therefore, the inclusion-exclusion principle yields $\sum_{j=0}^{n}(-1)^j\binom{n}{j}M_{n-j}$ as the number of perfect matchings of K_{2n} disjoint from G. Thus

$$P_n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{M_{n-j}}{M_n}.$$

Now $\lim_{n\to\infty} P_n$ can be computed by applying Lebesgue's dominated convergence theorem. Let $X = \{0, 1, 2, \ldots\}$, and define a measure μ on X by $\mu(\{j\}) = 1/j!$. Let $f_n\colon X \to \mathbb{R}$ be defined by

$$f_n(j) = \frac{(-1)^j n! M_{n-j}}{(n-j)! M_n} = (-1)^j \prod_{i=0}^{j-1} \frac{n-i}{2n-2i-1}.$$

Then $\lim_{n\to\infty} f_n(j) = (-1/2)^j$. Furthermore, $|f_n(j)| \le 1$, and the constant function 1 is integrable, since $\int_X 1\,d\mu = \sum_{j=0}^{\infty} 1/j! = e$. Therefore,

$$\lim_{n\to\infty} P_n = \lim \int_X f_n\,d\mu = \int_X \lim f_n\,d\mu = \sum_{j=0}^{\infty}\frac{(-1/2)^j}{j!} = e^{-1/2}.$$

Solved also by R. J. Chapman (U. K.), R. DiSario, J. Grossman, J. Labelle, D. Tenny, NCCU Problems Group, and the proposer.

Characterizations of the Medial Triangle

10588 [1997, 361]. *Proposed by Marcin Mazur, The University of Chicago, Chicago, IL.* Let $A_1A_2A_3$ be a triangle. For $i = 1, 2, 3$, let B_i be a point on side $A_{i+1}A_{i+2}$, where subscripts are taken modulo 3.

(**a**) Show that $|A_iB_{i+1}| + |B_iB_{i+1}| = |A_iB_{i+2}| + |B_iB_{i+2}|$ for $i = 1, 2, 3$ if and only if B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$.

(**b**) Show that $|A_iB_{i+1}| + |A_iB_{i+2}| = |B_iB_{i+1}| + |B_iB_{i+2}|$ for $i = 1, 2, 3$ if and only if B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$.