continus, *Tôhoku Mat. J.* 13 (1918) 300–303. Klee also noted that circular disks are *smooth* (i.e., possess a continuously differentiable parameterization) as well as rotund. For more on smooth tilings, see V. Klee, E. Maluta, and C. Zanco, Tiling with smooth and rotund tiles, *Fund. Math.* 126 (1986) 269–290; V. Klee and C. Tricot, Locally countable plump tilings are flat, *Math. Ann.* 277 (1987) 315–325; and P. M. Gruber, How well can space be packed with smooth bodies? Measure theoretic results, *J. London Math. Soc.* (2) 52 (1995) 1–14.

D. G. Larman, A note on the Besicovich dimension of the closest packing of sphere in R_n, *Proc. Cambridge Philos. Soc.* 62 (1966) 193–195 shows that, in the case of packing of circular disks in the plane, the uncovered set has Hausdorff dimension at least 1.03.

Solved also by G. E. Bredon, P. Budney, J. D. Clemens, J. Cobb, R. Holzsager, A. A. Jagers (The Netherlands), V. Klee, J. H. Lindsey II, O. P. Lossers (The Netherlands), R. Martin (Germany), L. E. Mattics, M. Misiurewicz, I. Namioka, O. Nanyes, C. G. Petalas & T. P. Vidalis (Greece), C. Popescu (Belgium), A. W. Schurle, J. H. Shapiro & T. L. McCoy, A. A. Tarabay & R. Barbara (Lebanon), and the Anchorage Math Solutions Group.

Random Perfect Matchings

10587 [1997, 361]. *Proposed by Joaquín Gómez Rey, Madrid, Spain.* Let K_{2n} be the complete graph on $2n$ vertices. Let P_n be the probability that two random perfect matchings of K_{2n} are disjoint. What is $\lim_{n\to\infty} P_n$?

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela. The limit is $e^{-1/2} \approx 0.60653$. The number of perfect matchings of K_{2n} is $M_n = (2n)!/(2^n n!)$. Given a perfect matching G of K_{2n} and a set J of j edges of G, there are M_{n-j} perfect matchings of K_{2n} containing J. Therefore, the inclusion-exclusion principle yields $\sum_{j=0}^{n}(-1)^j\binom{n}{j}M_{n-j}$ as the number of perfect matchings of K_{2n} disjoint from G. Thus

$$P_n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{M_{n-j}}{M_n}.$$

Now $\lim_{n\to\infty} P_n$ can be computed by applying Lebesgue's dominated convergence theorem. Let $X = \{0, 1, 2, \ldots\}$, and define a measure μ on X by $\mu(\{j\}) = 1/j!$. Let $f_n\colon X \to \mathbb{R}$ be defined by

$$f_n(j) = \frac{(-1)^j n! M_{n-j}}{(n-j)! M_n} = (-1)^j \prod_{i=0}^{j-1} \frac{n-i}{2n-2i-1}.$$

Then $\lim_{n\to\infty} f_n(j) = (-1/2)^j$. Furthermore, $|f_n(j)| \le 1$, and the constant function 1 is integrable, since $\int_X 1\,d\mu = \sum_{j=0}^{\infty} 1/j! = e$. Therefore,

$$\lim_{n\to\infty} P_n = \lim \int_X f_n\,d\mu = \int_X \lim f_n\,d\mu = \sum_{j=0}^{\infty}\frac{(-1/2)^j}{j!} = e^{-1/2}.$$

Solved also by R. J. Chapman (U. K.), R. DiSario, J. Grossman, J. Labelle, D. Tenny, NCCU Problems Group, and the proposer.

Characterizations of the Medial Triangle

10588 [1997, 361]. *Proposed by Marcin Mazur, The University of Chicago, Chicago, IL.* Let $A_1A_2A_3$ be a triangle. For $i = 1, 2, 3$, let B_i be a point on side $A_{i+1}A_{i+2}$, where subscripts are taken modulo 3.

(**a**) Show that $|A_iB_{i+1}| + |B_iB_{i+1}| = |A_iB_{i+2}| + |B_iB_{i+2}|$ for $i = 1, 2, 3$ if and only if B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$.

(**b**) Show that $|A_iB_{i+1}| + |A_iB_{i+2}| = |B_iB_{i+1}| + |B_iB_{i+2}|$ for $i = 1, 2, 3$ if and only if B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$.

Solution by the proposer. If B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$, then triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, so $|B_1B_2| = (1/2)|A_1A_2|$, $|B_2B_3| = (1/2)|A_2A_3|$, and $|B_3B_1| = (1/2)|A_3A_1|$. Hence

$$|A_1B_2| + |B_1B_2| = \frac{1}{2}|A_1A_3| + \frac{1}{2}|A_1A_2| = |B_1B_3| + |A_1B_3|,$$

and similarly for the other conditions of both parts.

(**a**) We prove that for any triangle $B_1B_2B_3$ there exists exactly one triangle $A_1A_2A_3$ such that $|A_iB_{i+1}|+|B_iB_{i+1}| = |A_iB_{i+2}|+|B_iB_{i+2}|$ for $i = 1, 2, 3$. This implies our assertion. Fix a triangle $B_1B_2B_3$, and suppose that for a triangle $A_1A_2A_3$ the conditions are satisfied. Let (i, j, k) be a permutation of $(1, 2, 3)$. Consider the hyperbola with foci B_j and B_k passing through B_i. Since $|A_iB_j| + |B_iB_j| = |A_iB_k| + |B_iB_k|$, the hyperbola passes through A_i. Write h_i for the part of the branch of the hyperbola passing through A_i that is on the opposite side of the line B_jB_k from B_i. Since B_j and B_k are the foci of the hyperbola, h_i is entirely contained in the union of all lines joining A_i and some point on the segment B_jB_k.

Now suppose that A is any point on h_1 different from A_1. (This A is a candidate for the vertex A_1 in a new triangle satisfying the conditions.) If A is inside triangle $B_2A_1B_3$, then the line from A through B_2 intersects h_3 in a point P that is on the opposite side of the line A_2A_3 from A_1, and if A is outside of $B_2A_1B_3$ then P is on the same side of A_2A_3 as A_1. (Point P is the candidate for point A_3 of the new triangle.) The same holds for the intersection Q of the line AB_3 with h_2 (the candidate for A_2 of the new triangle). Therefore, the line segment PQ does not pass through B_1. We conclude that A cannot be a vertex of a triangle that satisfies our requirements. A similar argument shows that no point A outside triangle $B_2A_1B_3$ can be a vertex of a triangle that satisfies our requirements. Thus $A_1A_2A_3$ is the only triangle for which the conditions hold.

(**b**) Let $a_k = (1/2)|A_iA_j|$, $b_k = |B_iB_j|$, and $x_j = a_j - |A_iB_j|$, where (i, j, k) is an even permutation of $(1, 2, 3)$. By hypothesis, $a_i + x_i + a_k - x_k = b_i + b_k$. Adding two of these equations and subtracting the third yields $b_i = a_i - x_j + x_k$, so

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 - 2a_ix_j - 2x_jx_k + 2a_ix_k \tag{1}$$

By the law of cosines we obtain $b_i^2 = (a_j+x_j)^2 + (a_k-x_k)^2 - 2(a_j+x_j)(a_k-x_k)\cos A_i$. Since $\cos A_i = a_j^2 + a_k^2 - a_i^2/2a_ja_k$ we get after simple transformations

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 + \frac{x_j}{a_j}(a_j^2 + a_i^2 - a_k^2) - \frac{x_k}{a_k}(a_k^2 + a_i^2 - a_j^2) + \frac{x_j}{a_j}\frac{x_k}{a_k}(a_j^2 + a_k^2 - a_i^2) \tag{2}$$

Let $z_i = x_i/a_i$. Comparing expressions (1) and (2) for b_i^2, we get

$$z_j(a_j + a_i - a_k) - z_k(a_k + a_i - a_j) + z_jz_k(a_j + a_k - a_i) = 0\,.$$

If one of the z_i's is 0, then all of them vanish. If they are all nonzero, then dividing by z_jz_k and adding all three equalities we get $a_1 + a_2 + a_3 = 0$, which is evidently false. Therefore, all the x_i's vanish and the assertion is proved.

Solved also by M. Vowe (Switzerland) and GCHQ Problems Group (U. K.).

Binary Expansions and kth Powers

10596 [1997, 456]. *Proposed by Paul Bateman, University of Illinois, Urbana, IL, and David Bradley, Simon Fraser University, Burnaby, BC, Canada.*

(**a**) Prove the identity

$$\sum_{j=0}^{2^{k-1}-1} (-1)^{k-1-\eta(j)}(y+j)^k = k!\cdot 2^{(k-1)(k-2)/2}(y + (2^{k-1}-1)/2),$$