NOTES

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Lexell's Theorem Via an Inscribed Angle Theorem

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We present a simple inscribed angle theorem in spherical geometry, and apply it to give a short proof of Lexell's theorem.

Theorem 1. *For any spherical triangle ABC inscribed in a fixed circular arc* Γ *with end-points* A, B, *the value of* $\angle C - (\angle A + \angle B)$ *is constant.*

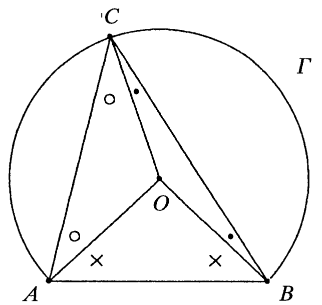

Figure 1

Proof: Let O be the center of the spherical cap determined by Γ. Then, since the base angles of a spherical isosceles triangle are equal, it follows easily from Figure 1 that

$$\angle C - (\angle A + \angle B) = \pm 2\angle OAB,$$

where the sign is $+$ if Γ is a minor arc, and $-$ otherwise. ■

Let $|ABC|$ denote the area of a spherical triangle ABC on the unit sphere. Then by Girard's formula, we have $|ABC| = \angle A + \angle B + \angle C - \pi$.

Theorem 2 (Lexell). *Let ABC be a spherical triangle on the unit sphere, and let* $\mathscr{H}$ *be the hemisphere bounded by the great circle AB and containing C. Then the locus of the point* $X \in \mathscr{H}$ *satisfying* $|ABX| = |ABC|$ *is the circular arc* A^*CB^*, *where* A^*, B^* *are the antipodal points of* A, B, *respectively.*

Proof: It suffices to show that $|ABX| = |ABC|$ for any point X on the circular arc A^*CB^* $(X \neq A^*, X \neq B^*)$, By theorem 1, we have $\angle A^*CB^* - (\angle CA^*B^* + \angle CB^*A^*) = \angle A^*XB^* - (\angle XA^*B^* + \angle XB^*A^*)$. Since $\angle A^*XB^* = \angle AXB$,

$\angle XA^*B^* = \pi - \angle XAB$, $\angle XB^*A^* = \pi - \angle XBA$, we have $\angle AXB + \angle BAX + \angle ABX = \angle ACB + \angle BAC + \angle ABC$. Hence, by Girard's formula, we have $|ABX| = |ABC|$. ∎

For a different proof of Lexell's theorem, see L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer-Verlag, Berlin, 1972, p. 23.

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A Characteristic Property of Differentiation

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We offer here a simple exercise in calculus with a flavor of functional analysis. The differentiation operator $D: f \to f'$ is a fundamental operator in calculus and it is interesting to consider what properties distinguish it from all other operators on functions. One important theorem says that if a differentiable function $f(x)$ has a relative minimum (or maximum) at $x = a$, then $f'(a) = 0$. As we shall see now, this property "almost" characterizes D.

Notation. For convenience we consider only polynomials. Let P be the set of all polynomials and let p_n, $n = 0, 1, \ldots,$ be the basic polynomials:

$$p_0(x) = 1, p_1(x) = x, \ldots, p_n(x) = x^n, \ldots .$$

When $\delta : P \to P$ is a linear operator, we denote its action on $p \in P$ by $\delta[p]$. Thus $\delta[p]$ is again a polynomial and its value at some number x is written as $\delta[p](x)$.

Theorem 1. *Let $\delta : P \to P$ be a linear operator. Then the following are equivalent:*

(i) *If p has a relative minimum at $x = a$, then $\delta[p](a) = 0$.*
(ii) $\delta = \delta[p_1]D$.

In particular, if $\delta[p_1] = p_0$, then $\delta = D$. (Here "minimum" can be replaced by maximum.")

Proof: The implication (ii) → (i) is immediate, so we focus on (i) → (ii). First we want to show that every linear operator on P has a convenient general form. By Taylor's formula, for any polynomial p and any number a:

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(a)}{k!}(x-a)^k.$$

The sum is finite and we write "∞" just for convenience. Applying δ to both sides (as polynomials of x, with a fixed) we obtain

$$\delta[p] = \sum_{k=0}^{\infty} \frac{p^{(k)}(a)}{k!}\delta\left[(x-a)^k\right] \tag{1}$$