fore, we conclude that diffusion in nature is governed by second-order elliptic partial differential operators. Theorem 2 (in a different form) originates from A. Kolmogorov. Some others contributed to it, providing modifications and extensions: comments and references can be found in [**1**, Chapter 5] and [**2**, Chapter XIII].

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A Weighted Mixed-Mean Inequality

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In [**4**], the author established the following inequality conjectured by Holland [**3**]. Unbeknownst to either of these parties, the same inequality had been earlier announced by Nanjundiah [**8**] without proof.

Theorem 1. *Let $x_1, x_2, \dots, x_n$ be positive real numbers. The arithmetic mean of the numbers*

$$x_1, \sqrt{x_1 x_2}, \dots, \sqrt[n]{x_1 x_2 \cdots x_n}$$

does not exceed the geometric mean of the numbers

$$x_1, \frac{x_1 + x_2}{2}, \dots, \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Here we prove the following weighted extension of Theorem 1.

Theorem 2. *Let $x_1, \dots, x_n, w_1, \dots, w_n$ be positive real numbers, and define $s_i = w_1 + \cdots + w_i$ for $i = 1, \dots, n$. Assume that*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots \geq \frac{w_n}{s_n}. \tag{1}$$

Then the weighted arithmetic mean of the numbers

$$x_1, x_1^{w_1/s_2} x_2^{w_2/s_2}, \dots, x_1^{w_1/s_n} x_2^{w_2/s_n} \cdots x_n^{w_n/s_n}$$

does not exceed the weighted geometric mean of the numbers

$$x_1, \frac{w_1}{s_2}x_1 + \frac{w_2}{s_2}x_2, \dots, \frac{w_1}{s_n}x_1 + \frac{w_2}{s_n}x_2 + \cdots + \frac{w_n}{s_n}x_n$$

when each mean is taken with weights $w_1/s_n, w_2/s_n, \dots, w_n/s_n$. In other words,

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{i} \frac{w_j}{s_i} x_j \right)^{w_i/s_n} \geq \sum_{j=1}^{n} \frac{w_j}{s_n} \prod_{i=1}^{j} x_i^{w_i/s_j}. \tag{2}$$

Equality holds if and only if $x_1 = \cdots = x_n$.

The constraint (1) might not be the weakest possible, but some constraint is definitely necessary; for example, one needs to have

$$\left(\frac{w_1}{s_1}\right)^{w_1}\left(\frac{w_2}{s_2}\right)^{w_2}\cdots\left(\frac{w_{n-1}}{s_{n-1}}\right)^{w_{n-1}} \ge \left(\frac{w_n}{s_n}\right)^{s_{n-1}}$$

or else (2) fails for $x_n \gg x_{n-1} \gg \cdots \gg x_1$. Preliminary calculations suggest that this condition might even be sufficient, but a proof seems difficult. Theorem 2 is asserted without any condition on the weights in [**1**, pp. 122–123]; of course the proof given there is incorrect.

The ingredients of the proof of Theorem 2 are the same as in [**4**], except that we use induction to simplify the computations; one may unravel the induction to obtain a proof that, in the case of equal weights, coincides with the proof in [**4**]. A different inductive proof of Theorem 1, using Lagrange multipliers, appears in [**5**].

Proof: We prove Theorem 2 by proving an analogue of Rado's inequality [**2**, Theorem 60] in this setting. Namely, if L_n and R_n denote the left and right sides of (2), we prove that

$$\left(\frac{L_n}{R_n}\right)^{s_n} \ge \left(\frac{L_{n-1}}{R_{n-1}}\right)^{s_{n-1}} \tag{3}$$

for $n > 1$. We note in passing that a similar argument gives an analogue of Popoviciu's inequality [**9**]:

$$s_n(L_n - R_n) \ge s_{n-1}(L_{n-1} - R_{n-1}).$$

Unraveling (3), we see that it is equivalent to

$$\left(\sum_{j=1}^{n}\frac{w_j}{s_n}x_j\right)^{w_n}\left(\sum_{j=1}^{n-1}\frac{w_j}{s_{n-1}}\prod_{i=1}^{j}x_i^{w_i/s_j}\right)^{s_{n-1}} \ge \left(\sum_{j=1}^{n}\frac{w_j}{s_n}x_j^{w_j/s_j}\prod_{i=1}^{j-1}x_i^{w_i/s_j}\right)^{s_n}. \tag{4}$$

We prove this inequality in two steps. First, we observe that

$$\sum_{j=1}^{n-1}\frac{w_j}{s_{n-1}}\prod_{i=1}^{j}x_i^{w_i/s_j} = \sum_{j=1}^{n}\left[\frac{w_js_n - w_ns_j}{s_{n-1}s_n}\prod_{i=1}^{j}x_i^{w_i/s_j} + \frac{s_{j-1}w_n}{s_{n-1}s_n}\prod_{i=1}^{j-1}x_i^{w_i/s_{j-1}}\right];$$

since $w_js_n \ge w_ns_j$ by (1), we may apply the weighted arithmetic-mean, geometric-mean inequality to each summand on the right side and obtain

$$\sum_{j=1}^{n-1}\frac{w_j}{s_{n-1}}\prod_{i=1}^{j}x_i^{w_i/s_j} \ge \sum_{j=1}^{n}\frac{w_j}{s_n}x_j^{\frac{w_js_n-w_ns_j}{s_js_{n-1}}}\prod_{i=1}^{j-1}x_i^{\frac{w_is_n}{s_js_{n-1}}}. \tag{5}$$

Second, we apply Hölder's inequality to get

$$\left(\sum_{j=1}^{n}\frac{w_j}{s_n}x_j^{\frac{w_js_n-w_ns_j}{s_js_{n-1}}}\prod_{i=1}^{j-1}x_i^{\frac{w_is_n}{s_js_{n-1}}}\right)^{s_{n-1}/s_n}\left(\sum_{j=1}^{n}\frac{w_j}{s_n}x_j\right)^{w_n/s_n} \ge \sum_{j=1}^{n}\frac{w_j}{s_n}x_j^{w_j/s_j}\prod_{i=1}^{j-1}x_i^{w_i/s_j}. \tag{6}$$

Together, (5) and (6) imply (4), and (2) now follows by induction on n (since equality vacuously holds for $n = 1$). The equality condition also follows by induction: if equality holds in (2), then equality in (3) forces $x_1 = \cdots = x_{n-1}$ by hypothesis, and equality in (6) forces $x_1 = x_n$.

We mention here three ways in which Theorem 2 can be extended easily. First, one can replace the arithmetic and geometric means by the r-th and s-th power means, respectively, for any $r > s$; the corresponding analogue of Theorem 1 is formulated in [**6**]. Recall that for $r \neq 0$, the r-th power mean of $x_1, \dots, x_n$ with weights $w_1, \dots, w_n$ is given by

$$\left(\frac{w_1}{s_n} x_1^r + \frac{w_2}{s_n} x_2^r + \cdots + \frac{w_n}{s_n} x_n^r \right)^{1/r}.$$

Extending by continuity to $r = 0$ yields the weighted geometric mean. The statement of the inequality then becomes

$$\left(\sum_{i=1}^{n} \frac{w_i}{s_n} \left(\sum_{j=1}^{i} \frac{w_j}{s_i} x_j^r \right)^{s/r} \right)^{1/s} \geq \left(\sum_{j=1}^{n} \frac{w_j}{s_n} \left(\sum_{i=1}^{j} \frac{w_i}{s_j} x_i^s \right)^{r/s} \right)^{1/r},$$

the Rado and Popoviciu-type inequalities become

$$s_n \left(L_n^k - R_n^k \right) \geq s_{n-1} \left(L_{n-1}^k - R_{n-1}^k \right) \quad k = r, s,$$

and the proofs carry over upon replacing the weighted arithmetic-mean, geometric-mean inequality by the weighted power mean inequality, and Hölder's inequality by Minkowski's inequality [**2**, Theorem 24]. This last inequality appears to hold for all $k \in [s, r]$, but I do not have a proof.

Second, one can prove a analogue of Theorem 2 for Hermitian matrices, using the arithmetic and harmonic means, following Mond and Pečarić [**7**], who proved such an analogue of Theorem 1 using a matricial Minkowski inequality.

Third, one may use a straightforward limiting argument to deduce the following continuous analogue of Theorem 2. We leave the formulation of the corresponding power mean generalization to the reader.

Theorem 3. *Let* $f(x)$ *and* $w(x)$ *be continuous positive-valued functions on* $[0, 1]$, *and let* $W(x) = \int_0^x w(t)\,dt$. *Assume that* $w(x)/W(x)$ *is nondecreasing on* $(0, 1]$. *Then*

$$\int_0^1 \exp\left(\frac{w(y)}{W(1)} \log \int_0^y \frac{w(x)}{W(y)} f(x)\,dx \right) dy \geq \int_0^1 \frac{w(y)}{W(1)} \exp\left(\int_0^y \frac{w(x)}{W(y)} \log f(x)\,dx \right) dy.$$

Finally, we use Theorem 2 to generalize a well-known inequality of Carleman: for a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers with $\sum a_n < \infty$,

$$\sum_{k=1}^{\infty} (a_1 \cdots a_k)^{1/k} < e \sum_{k=1}^{\infty} a_k.$$

It was observed in [**3**] and in [**8**] that this inequality follows from Theorem 1. We refine this observation slightly to obtain a weighted version of Carleman's inequality. Surprisingly (to the author, at least), the constant on the right side does not depend on the weights!

Theorem 4. *Let* $w_1, w_2, \dots$ *be a sequence of positive real numbers, and define* $s_i = w_1 + \cdots + w_i$ *for* $i = 1, 2 \dots$ *. Assume that*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots.$$

Then for any sequence $a_1, a_2, \dots$ *of positive real numbers with* $\sum_k w_k a_k < \infty$,

$$\sum_{k=1}^{\infty} w_k a_1^{w_1/s_k} \cdots a_k^{w_k/s_k} < e \sum_{k=1}^{\infty} w_k a_k.$$

Proof: Taking $x_k = a_k$ for $k = 1, \dots, n$ in Theorem 2, we obtain

$$\sum_{k=1}^{n} \frac{w_k}{s_n} a_1^{w_1/s_k} \cdots a_k^{w_k/s_k} \le \prod_{k=1}^{n} \left(\sum_{i=1}^{k} \frac{w_i}{s_i} a_i \right)^{w_k/s_n}.$$

Of course $\sum_{i=1}^{k} w_i a_i \le \sum_{i=1}^{n} w_i a_i$, and so

$$\sum_{k=1}^{n} w_k a_1^{w_1/s_k} \cdots a_k^{w_k/s_k} \le \frac{s_n}{s_1^{w_1/s_n} \cdots s_n^{w_n/s_n}} \sum_{k=1}^{n} a_k.$$

In addition, using partial summation and the bound $\log x < x - 1$ for $x > 0$, we get

$$\begin{aligned} \frac{s_n}{s_1^{w_1/s_n} \cdots s_n^{w_n/s_n}} &= \exp \sum_{k=1}^{n} \frac{w_k}{s_n} (\log s_n - \log s_k) \\ &= \exp \sum_{k=1}^{n-1} \frac{s_k}{s_n} (\log s_{k+1} - \log s_k) \\ &< \exp \sum_{k=1}^{n-1} \left(\frac{s_{k+1}}{s_n} - \frac{s_k}{s_n} \right) = \exp\left(1 - \frac{s_1}{s_n}\right) < e. \end{aligned}$$

Thus we have the desired inequality except with n in place of ∞, but taking $n \to \infty$ gives what we want. ■

Again, one can easily state and prove power mean and continuous analogues, and again the conditions on the weights are probably not the weakest possible.

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