An Elegant Continued Fraction for π

L. J. Lange

The regular continued fraction for π begins as follows [**3**, p. 23]:

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{14} + \cdots. \quad (1)$$

There is no known regularity to the partial denominators in (1) and the only known means to obtain them is to compute them one-by-one from a known decimal approximation for π. Lord Brouncker (1620–1686), the first president of the Royal Society of London, gave (without proof around 1659) the first recorded infinite continued fraction [**3**, p. 2]:

$$\frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \frac{9^2}{2} + \frac{11^2}{2} + \frac{13^2}{2} + \cdots. \quad (2)$$

In 1775, according to [**1**, p. 131], Euler gave a proof of the validity of (2) by showing that

$$\arctan x = \frac{x}{1} + \frac{1^2x^2}{3 - x^2} + \frac{3^2x^2}{5 - 3x^2} + \frac{5^2x^2}{7 - 5x^2} + \cdots \quad (3)$$

is equivalent to the power series representation

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad -1 \le x \le 1.$$

Brouncker's result can be obtained by setting $x = 1$ in (3).

The following continued fraction expansion for the principal branch of the analytic function $\arctan z$, valid for all z in the complex plane not on the imaginary axis from i to $+i\infty$ and from $-i$ to $-i\infty$, is well known [**3**, p. 202]:

$$\arctan z = \frac{z}{1} + \frac{1^2z^2}{3} + \frac{2^2z^2}{5} + \frac{3^2z^2}{7} + \frac{4^2z^2}{9} + \cdots. \quad (4)$$

Setting $z = 1$ in (4) leads to

$$\frac{\pi}{4} = \frac{1}{1} + \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \frac{4^2}{9} + \cdots. \quad (5)$$

Although they are not formulas for π itself, the classical continued fractions (2) and (5) are attractive because of the simple expressions for all of their partial numerators and denominators. Our contribution is the following continued fraction for π itself, whose partial numerators and denominators are easily described and remembered. Though the tools to derive it have long been available, to our knowledge, this formula has not yet appeared in the literature.

Theorem 1.

$$\pi = 3 + \frac{1^2}{6} + \frac{3^2}{6} + \frac{5^2}{6} + \frac{7^2}{6} + \frac{9^2}{6} + \frac{11^2}{6} + \frac{13^2}{6} + \cdots. \quad (6)$$

Proof: We think it is of interest to show in several different ways that (6) is valid. Perron [**5**, p. 35] gives the following representation, which he attributes to Stieltjes:

$$x + \frac{1^2 - n^2}{2x} + \frac{3^2 - n^2}{2x} + \frac{5^2 - n^2}{2x} + \cdots = 4 \cdot \frac{\Gamma\left(\frac{x+3+n}{4}\right)\Gamma\left(\frac{x+3-n}{4}\right)}{\Gamma\left(\frac{x+1+n}{4}\right)\Gamma\left(\frac{x+1-n}{4}\right)}, \tag{7}$$

where $x > 0$ and $1 > n^2 > -\infty$. Setting $n = 0$ in (7) gives

$$x + \frac{1^2}{2x} + \frac{3^2}{2x} + \frac{5^2}{2x} + \cdots = 4 \cdot \frac{\Gamma\left(\frac{x+3}{4}\right)\Gamma\left(\frac{x+3}{4}\right)}{\Gamma\left(\frac{x+1}{4}\right)\Gamma\left(\frac{x+1}{4}\right)}, \tag{8}$$

which is a formula also obtained by Ramanujan and Preece according to Perron [**5**, p. 36]. To obtain (6) we have only to substitute $x = 3$ in (8) and employ the properties $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$ of the Γ-function. It is surprising that apparently Ramanujan either was not aware of, or else did not choose to record this result. To show how we really arrived at (6) the first time, we need the following result [**5**, Satz 1.13, p. 28] relating to what are known as *Bauer-Muir transformations* of continued fractions; see [**4**].

Theorem 2. (*a*) *If both continued fractions*

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \quad \textit{and}$$

$$b_0 + r_0 + \frac{\varphi_1}{b_1 + r_1} + \frac{a_1 \frac{\varphi_2}{\varphi_1}}{b_2 + r_2 - r_0 \frac{\varphi_2}{\varphi_1}} + \frac{a_2 \frac{\varphi_3}{\varphi_2}}{b_3 + r_3 - r_1 \frac{\varphi_3}{\varphi_2}} + \cdots,$$

where $\varphi_\nu = a_\nu - r_{\nu-1}(b_\nu + r_\nu)$, *have positive elements and if both converge, then they have the same value.* (*b*) *If the first continued fraction has positive elements and it converges and if* $r_\nu \geq 0$ *from a certain* ν *on, then the second continued fraction also converges and it has the same value as the first.*

The second continued fraction in Theorem 2 is called the *Bauer-Muir transform* of the first one. On page 35 of [**5**] is the expansion

$$z \cot \frac{\pi z}{4} = 1 + \frac{1^2 - z^2}{2} + \frac{3^2 - z^2}{2} + \frac{5^2 - z^2}{2} + \frac{7^2 - z^2}{2} + \cdots, \tag{9}$$

which is valid for all complex z. If we apply Theorem 2 to this continued fraction with $z = x \in (-1, 1)$ and

$$a_n = (2n-1)^2 - x^2, \quad b_n = 2, \quad r_n = 2n - 1, \quad \varphi_n = 4 - x^2,$$

we obtain

$$x \cot \frac{\pi x}{4} = \frac{2^2 - x^2}{3} + \frac{1^2 - x^2}{6} + \frac{3^2 - x^2}{6} + \frac{5^2 - x^2}{6} + \frac{7^2 - x^2}{6} + \cdots. \tag{10}$$

Taking the limit of both sides of (9) as $z \to 0$ gives Brounker's result (2), and taking the limit of both sides of (10) as $x \to 0$ leads to (6) upon taking reciprocals.

It would be nice if the speed of convergence of (6) was in accordance with its beauty, but unfortunately this is not the case. In support of this slowness assertion the 100th approximant of (6) rounds to 3.14159241, whereas both π and the 4th approximant of its regular continued fraction expansion (1) round to 3.14159265. If the expansions (2) and (5) are used to approximate π, the 11th approximant of (5) gives 3.14159265 as an approximation, but Brouncker's continued fraction (2) converges so slowly that its 1000th approximant leads to the poor estimate of 3.14259165 for π. As another source of information about π, we recommend to the reader the recent book [**2**].

Addendum:. The formula (6) was used as a logo for the conference on continued fractions that was held at the University of Missouri-Columbia in late May 1998. At this conference D. Bowman of the University of Illinois mentioned in a personal conversation that he had another approach to deriving (6). Bowman starts with the result

$$\frac{\pi - 3}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)} = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{k} + \frac{1}{k+1} - \frac{4}{2k+1} \right) \tag{11}$$

and then makes use of the fact that for $a_k \neq 0$ the series $\sum_{k=1}^{\infty} (-1)^{k-1}/a_k$ and the continued fraction

$$\frac{1}{a_1} + \frac{a_1^2}{a_2 - a_1} + \frac{a_2^2}{a_3 - a_2} + \frac{a_3^2}{a_4 - a_3} + \cdots \tag{12}$$

are equivalent, that is, the nth partial sum of the series and the nth approximant of the continued fraction are equal. This connection between series and continued fractions can be derived easily from a result of Euler (see [**5**, p. 17] or [**3**, p. 37]), or it can be proved directly by induction. After replacing a_k by $2k(2k+1)(2k+2)$ in (12) and calculating $a_{k+1} - a_k = 24(k+1)^2$, we are led to the representation (6) through a simple cancellation process that preserves the equivalence of the continued fractions involved. Bowman mentioned that his approach to verifying (6) gives as a welcome by-product some immediate truncation error information. Because of the series-continued fraction equivalence and the alternating nature of the first series in (11), we have $|\pi - f_n| \leq 1/((n+1)(n+2)(2n+3))$, where f_n is the nth approximant of (6).

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University of Missouri, Columbia, MO 65211
jerry@math.missouri.edu