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THE EVOLUTION OF...

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Riemann's Dissertation and Its Effect on the Evolution of Mathematics

Detlef Laugwitz

Translated from the German by Abe Shenitzer†

A short account of the contents of the dissertation. Riemann's doctoral dissertation of 1851 is titled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (*Foundations for a general theory of functions of a variable complex quantity*) [**1**, 3–43]. It is of modest size. In discussing it we use modern terms.

Riemann defines holomorphic functions as complex single-valued functions on Riemann surfaces satisfying the Cauchy-Riemann differential equations. Riemann also worked with functions that were holomorphic except for finite poles in $\mathbb{C}$. Such meromorphic functions are viewed as conformal mappings between two Riemann surfaces. We must always think of the complex plane as extended by the addition of the point ∞ (as the Riemann complex number sphere or as a complex projective straight line).

Functions must be thought of not as given by expressions but as *determined* (to within arbitrary constants) *by the positions and nature of their singularities*. This leads to the question of the construction of functions with prescribed properties on a given Riemann surface. Here the topology of the surface is of decisive importance. The surface T is decomposed by means of n crosscuts into a system of m simply connected surface pieces. The number $n - m$, which is independent of the manner of decomposition, is called the order of connectivity of T [**1**, 10–11]; incidentally, in modern terms, this number is equal to the negative of the Euler characteristic of T.

In order to construct appropriate functions on T, Riemann uses a variational principle. (He called it later the Dirichlet principle because he came to know similar procedures in Dirichlet's lectures, and the historically unjustified name stuck.) First T is made into a simply connected surface T^* by means of crosscuts. Then, subject to suitable boundary conditions, the integral

$$\int\left[(u_x - v_y)^2 + (u_y + v_x)^2\right] dx\, dy$$

†*Translator's note.* Reprinted from "*Bernhard Riemann 1826–1866: Turning Points in the Conception of Mathematics*," by Detlef Laugwitz, Translated by Abe Shenitzer. Copyright 1999 Birkhäuser. This article is an excerpt (Section 1.2.2, pp. 108–110 and Section 1.2.5, pp. 124–130) from the author's book *Bernhard Riemann*, published by Birkhäuser Verlag in 1996. References such as Article 20 or §20 are to sections of Riemann's dissertation.

is minimized on this surface. If there are singularities to be taken into consideration, then the integral is somewhat modified. With the possible exception of the boundary of T^*, the pair of functions u, v associated with the minimum is a holomorphic function $f = u + iv$. It should be noted that the functional values on the two edges of a crosscut need not coincide; jumps ("periods") may occur.

The paper ends with an application of these methods to the Riemann Mapping Theorem. This theorem asserts that in certain cases the topological equivalence of two surfaces or regions implies their conformal equivalence, i.e., the existence of a conformal mapping between them. Here the theorem is first stated for regions in the complex plane that are homeomorphic to a circular disk.

We will examine the individual key words while considering further developments in the work of Riemann and others.

We explain briefly, in modern terms, the form of inference Riemann learned from Dirichlet. Let $I(\varphi, \psi)$ be the integral of $\varphi_x \psi_x + \varphi_y \psi_y$ over a region G and let $J(\varphi) = I(\varphi, \varphi)$. Let η be a function that vanishes on the boundary ∂G of G.

$$J(\varphi + t\eta) = J(\varphi) + 2tI(\varphi, \eta) + t^2 J(\eta)$$

implies that if $J(\varphi) \leq J(\varphi + t\eta)$ is to hold for all t, then we must have $I(\varphi, \eta) = 0$. Put $\Delta\varphi = \varphi_{xx} + \varphi_{yy}$. Our last result, the vanishing of η on ∂G, and the Gauss integral formula (Gauss' theorem) imply that

$$0 = \int_{\partial G} (\varphi_x \eta \, dy - \varphi_y \eta \, dx) = \int_G (\Delta\varphi)\eta \, dF + I(\varphi, \eta) = \int_G (\Delta\varphi)\eta \, dF.$$

Since this holds for every η, it follows that $\Delta\varphi = 0$. In other words, a function that minimizes $J(\varphi)$ is a solution of $\Delta\varphi = 0$. To be sure, the argument does not prove the *existence* of such a function, and this elicited justified criticism.

It is relatively easy to prove the uniqueness of the solution of the boundary-value problem. If ψ were another solution, then $\eta = \varphi - \psi$ would vanish on ∂G. Moreover,

$$J(\varphi) = J(\psi) + 2I(\psi, \eta) + J(\eta)$$

and

$$I(\psi, \eta) = \int_{\partial G} \eta(\psi_x \, dy - \psi_y \, dx) - \int_G (\Delta\psi)\eta \, dF = 0.$$

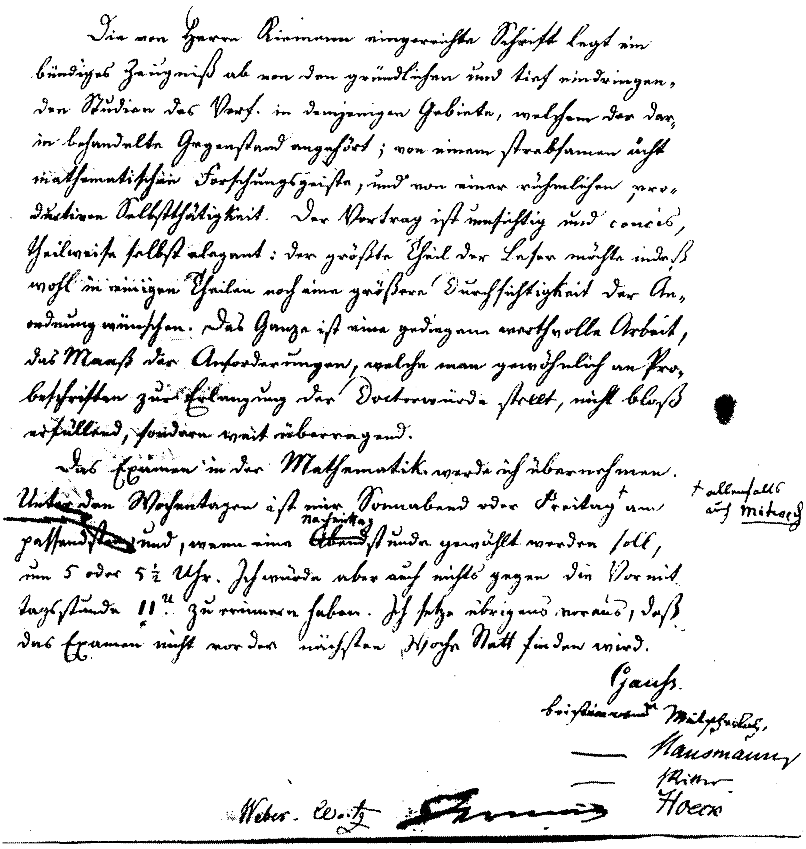
But then

$$J(\varphi) = J(\psi) + J(\eta) \geq J(\psi).$$

In view of the minimality of $J(\varphi)$, the inequality sign in the last expression must be replaced by an equality sign. But then $J(\eta) = 0$, i.e., $\eta_x = \eta_y = 0$. Since $\eta = 0$ on ∂G, it follows that $\eta = 0$, and therefore $\psi = \varphi$ throughout G.

The effect of the dissertation. Today we are inclined to regard Riemann's dissertation as one of the most important achievements of 19th-century mathematics, but its immediate effect was rather slight. We saw that in the second part of Article 20 Riemann himself emphasized just one principle, namely the determination of a function by as few data as possible and the elimination of expressions as definitions of functions. Given its vague formulation, this principle must have struck his contemporaries as neither new nor interesting. Riemann was as restrained in his statement as he was in the specification of his sources.

The first person who had to read the paper carefully was the referee for the Göttingen faculty, that is, Gauss. His report read as follows: "The paper submitted by Herr Riemann is a concise testimony to its author's thorough and penetrating studies of the area to which the subject treated therein belongs; of a diligent and ambitious, truly mathematical spirit of investigation, and of praiseworthy and fertile independence. The report is prudent and concise, and in places even elegant; nevertheless, most readers might well wish for even greater transparency of arrangement in some of the parts. Taken in its entirety, it is a solid and valuable work which not only meets the requirements usually set for test papers for the attainment of the doctorate but exceeds them by far."



Die von Herrn Riemann eingereichte Schrift legt ein bündiges Zeugniß ab von den gründlichen und tief eindringenden Studien des Verf. in demjenigen Gebiete, welchem der darin behandelte Gegenstand angehört; von einem strebsamen ächt mathematischen Forschungsgeiste, und von einer rühmlichen productiven Selbstthätigkeit. Der Vortrag ist umsichtig und concis, theilweise selbst elegant: der größte Theil der Leser möchte indeß wohl in einigen Theilen noch eine größere Durchsichtigkeit der Anordnung wünschen. Das Ganze ist eine gediegene werthvolle Arbeit, das Maaß der Anforderungen, welche man gewöhnlich an Probeschriften zur Erlangung der Doctorwürde stellt, nicht bloß erfüllend, sondern weit überragend.

Das Examen in der Mathematik werde ich übernehmen. Unter den Wochentagen ist mir Donnerstag oder Freitag + am passendsten, wenn eine Nachmittagsstunde gewählt werden soll, um 5 oder 5½ Uhr. Ich würde aber auch nichts gegen die Vormittagsstunde 11 U. zu erinnern haben. Ich setze übrigens voraus, daß das Examen nicht vor der nächsten Woche Statt finden wird.

+ allenfalls auch Mittwoch

Gauss.

beistimmend Mitscherlich, Hausmann, Ritter, Hoeck, Weber. Waitz

Figure 1. Gauss' testimonial on Riemann's dissertation

If one has a certain amount of experience with evaluations and forgets for a moment that here the *princeps mathematicorum* is writing about a person destined to become probably the most distinguished of his students, then one gets the following impression. The referee recognizes that the author has penetrated deep into a highly specialized field and has done this with great diligence, independently, and without the referee having to suggest the topic to him. There is no mention of the author's new ideas, of the solution of problems, or of new methods, but it is recognized that he may well be showing signs of independent research activity. The presentation is terse, elegant only in spots, and on the whole not clear enough. An objective reader must wonder what was the basis for the "Doktorvater's" (doctoral adviser's) very positive overall evaluation stated in the last sentence. Riemann wrote to his brother:

> When I visited Gauss he had not yet read my paper, but he told me that for years he had been preparing a paper (and is occupied with this right now) whose subject is the same, or partly the same, as the one I am treating

(Incidentally, this passage was quoted by Schering in his memorial address in 1866 [**2**, 835].) So far, no one has been able to find any indication that Gauss had discussed with Riemann the contents of his paper or had given him any hints or suggestions. Riemann would have reported such things. After all, he mentioned the rather disappointing conversation with Gauss which comes down, more or less, to this: right now I happen to be writing on a related topic, but your paper has not interested me enough that I should immediately and eagerly plunge into it.

Some (e.g., Remmert [**6**, Band 2, 158]) think that the old Gauss was "chary of praise" ("lobkarg"). But what argues against this is the fact that a few years earlier he had praised young Eisenstein to the skies. We will make no guesses about the great Gauss' admittedly baffling behavior toward Riemann.

We summarize the essential mathematical concerns that originated in Riemann's dissertation.

(1) The idea of a Riemann surface. Here, for the first time, the domain of definition of a function becomes one of the data that determine it. The complex plane is compactified by the addition of a single point ∞, the Riemann surfaces over it are precisely defined, the connectivity number is introduced and recognized as a topological invariant. (Complex) analysis is carried out not locally but on manifolds, which are compact in the case of algebraic functions. Local representability (by power series) is proved but is of secondary importance.

(2) In addition to poles, branch points are recognized as characteristic types of singularities, and the local series expansions in terms of (negative or fractional) powers are rigorously justified (Article 13/14, [**1**, 24–27]).

(3) The existence (together with the continuity) of $f'(z)$ is equivalent to the Cauchy–Riemann differential equations (together with the continuity of the occurring partial derivatives) and to the conformal character of f. It is also equivalent to the local expandibility, which implies the existence of all derivatives. (Holomorphic or analytic functions.)

(4) The transformation of surface integrals into line integrals is a tool for proving theorems (Articles 7–12, [**1**, 12–24]) of the "Cauchy type."

(5) The ("Dirichlet") principle of the existence of a function that minimizes a surface integral is used to solve boundary-value problems by means of holomorphic functions.

(6) The Riemann Mapping Theorem is a consequence of (5).

The response of contemporaries was amazingly slight; hardly any of the more than 500 titles in Purkert's list covering the period from 1851 to 1891 ([**2**, 869–895]) and relevant to Riemann's dissertation appeared before his death. This is all the more surprising if we keep in mind that two of Riemann's papers that presented the ideas of his dissertation in greater detail and applied them to the solution of problems appeared in 1857. Things were no different when it comes to textbooks. For example, Heinrich Weber's *Elliptische Functionen* of 1891 contains nothing relating to Riemann. Thus one can hardly speak of a significant impact of Riemann's ideas during his lifetime and in the first 25 years after his death. In the subsequent sections we will examine the question of the very special directions in which Riemann influenced research and the question of which elements of his essential ideas failed initially to attract attention.

Let us return to the year of the composition of the dissertation. Jacobi died on 18 February 1851. Dirichlet pushed Riemann in another direction, which led to his habilitation paper on trigonometric series. Representatives of the algorithmic direction could hardly be expected to approve of Riemann's dissertation. Eisenstein died on 11 October 1852 and Weierstrass had not yet appeared on the scene. The French mathematicians, whose contributions were not explicitly acknowledged in the dissertation, could at best be expected to recognize the concept of a Riemann surface as new. At the same time, they viewed it as too complicated and superfluous. Moreover, Cauchy's students soon got used to working with complex functions in the complex plane in much the same way as Cauchy, who had used complex formulations for his integral theorems and for his method of residues as early as 1831. They must have regarded the method of real partial differential equations as a backward step. At the time doubly periodic functions were in fashion, and they could be dealt with without the use of Riemann surfaces.

Of course, in time the six previously listed key issues associated with the dissertation exerted a powerful effect. What follows is a survey describing this effect.

The effect of (6) was later especially notable in applied mathematics. For a disk, the first boundary-value problem for the potential equation $u_{xx} + u_{yy} = 0$ is solved by the Poisson integral, which expresses the function u in terms of its boundary values. Since the differential equation is invariant under conformal mappings, we obtain a solution of this problem for any simply connected region bounded by a curve by mapping the disk conformally onto this region. But this is just an existence statement, and Riemann's theorem does not directly yield a formula representing the solution. Such representations were eventually obtained for regions of practical importance by H. A. Schwarz, E. B. Christoffel, and others.

The mapping theorem became effective in many respects independently of applications and of the other objectives and contents of the dissertation. It is an instance of Riemann's novel view of mathematics. For one thing, it illustrates the fruitfulness of the notion that functions are simply mappings. For another, it is a global proposition; all Gauss could prove was the conformal equivalence of small pieces of surfaces. Finally it was one of the deeper existence theorems to emerge after Cauchy's existence theorems about solutions of differential equations. For adherents of algorithms this was an unusual type of proposition; indeed, *they* took note of transformations only if they were associated with effective formulas. It is also noteworthy that the theorem shows that the theory of functions on a simply connected region with boundary is completely independent of the special choice of region. When investigating a special class of functions we can choose a convenient special region, say the upper halfplane.

Riemann's sketch of a proof in §21 is cryptic, and not just because of his use of the Dirichlet principle. Efforts to fully justify the idea of his proof failed. Given the importance of the theorem for applications, this failure stimulated attempts to develop new methods of proof. These remarks also apply to the uniformization theorem, which generalizes Riemann's mapping theorem. The geometric formulation promoted the acceptance of the notion of a Riemann surface. Riemann himself spoke [**1**, 40] of "geometric clothing" ("geometrische Einkleidung") used for "illustration and more convenient wording" (zur "Veranschaulichung und bequemeren Fassung"), formulations hardly ever encountered elsewhere in his writings. The use of complex methods for the computation of definite integrals opened up a new field for the applicability of complex function theory, and that is why complex analysis became a fixed component of the mathematical education of physicists and engineers. As for mathematics itself, the question of admissible boundaries of simply connected regions provided essential impulses for the evolution of point set theory.

For the effects of the dissertation in the first fifty years after Riemann, see [**5**]. For later developments see [**6**, Band 2, 157–163]. We recommend [**3**] and especially [**4**], a book saturated with Riemann's style of thinking. It is safe to say that, even had Riemann's dissertation consisted of just the mapping theorem, its influence would ultimately have been considerable.

The effect of (5) was unexpected. Riemann's justification of the existence of a minimal solution is inadequate. This was noted by Weierstrass, whose 1870 criticism was devastating and seemed to destroy the very basis of Riemann's justification of complex analysis. But this had also very positive consequences.

One consequence was that people tried, successfully, to prove the relevant results without using the Dirichlet principle. Actually they would have tried to find such proofs regardless of doubts about this principle. Such attempts reflect the wish to construct complex function theory in a "purely complex" way and to avoid the use of tools from real analysis, functions u and v of two real variables x and y. This too was achieved. Incidentally, this does not signify the rejection of Riemann's development of function theory. In view of its conceptual basis, it is closer to our way of thinking than is, say, the Weierstrass approach.

Another consequence of the criticism directed at Riemann's justification of the Dirichlet principle was even more important than the first one. Since there were no counterexamples and the principle itself was believable, people felt that it must be provable. Hilbert obtained a proof after 1900, and in doing so developed the so-called direct methods of the calculus of variations, which avoid the detour through the partial differential equations associated with the variational problem. One begins instead with a sequence of functions for which the values of the integral, or more generally of the functional, to be minimized approximate the infimum. One must show that the space of admissible functions has a compactness property which justifies the conclusion that a subsequence converges to a function for which the functional takes on its minimum. In this way a method was developed that not only saved the Dirichlet principle but has progressively become more important in the 20th century.

But let us go back briefly to the attempts to avoid the Dirichlet principle. Much was achieved by H. A. Schwarz and C. Neumann. As for the mapping theorem, the conclusive result was obtained independently by Poincaré and by Koebe in 1907. It asserts that every simply connected Riemann surface is holomorphically equivalent to one of following three surfaces: $\mathbb{C} \cup \{\infty\}$ (the number sphere or complex projective straight line), $\mathbb{C}$ (the number plane or complex straight line), or the open disk $|z| < 1$. The key that leads one to this group of problems in the literature is the uniformization theorem. This problem and its easy-to-formulate answer were almost obvious to Riemann, but half a century was needed to obtain it.

We do not know whether Riemann expected a stronger response. After all, he did say

> However, we now refrain from the realization of this theory...for we rule out, at present, consideration of an expression of a function

He set aside for a few years the task of investigating concrete functions and classes of functions, and tackled it in connection with lectures devoted to these matters. Of course, this did not happen during his first year as university instructor.

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Snow and Ice and Numbers

It seems necessary to explain my claustrophobia to him.

"Do you know what the foundation of mathematics is?" I ask. "The foundation of mathematics is numbers. If anyone asked me what makes me truly happy, I would say: numbers. Snow and ice and numbers. And do you know why?"

He splits the claws with a nutcracker and pulls out the meat with curved tweezers.

"Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child discovers a sense of longing, and do you know what the mathematical expression is for longing?"

He adds cream and several drops of orange juice to the soup.

"The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produce rational numbers. And human consciousness doesn't stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers."

He warms French bread in the oven and fills the pepper mill.

"It's a form of madness. Because the irrational numbers are infinite. They can't be written down. They force human consciousness out beyond the limits. And by adding irrational numbers to rational numbers, you get real numbers."

I've stepped into the middle of the room to have more space. It's rare that you have a chance to explain yourself to a fellow human being. Usually you have to fight for the floor. And this is important to me.

"It doesn't stop. It never stops. Because now, on the spot, we expand the real numbers with imaginary square roots of negative numbers. There are numbers we can't picture, numbers that normal human consciousness cannot comprehend. And when we add the imaginary numbers to the real numbers, we have the complex number system. The first number system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast, open landscape. The horizons. You head toward them and they keep receding. That is Greenland, and that's what I can't be without! That's why I don't want to be locked up."

Smilla's Sense of Snow, by Peter Høeg, translated by Tiina Nunnally
Dell Publishing, New York, 1994, pp. 121–122

Contributed by Evan J. Romer, Windsor, NY