10737. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.* Let m and n be positive integers with $n \geq 2m$, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that

$$a_n < m + \frac{1}{2m}\left(\sum_{i=1}^{m}\binom{n}{2i}\binom{2i}{i}\right).$$

Show that there exist two different n-tuples $(\epsilon_1, \ldots, \epsilon_n)$ and $(\delta_1, \ldots, \delta_n)$, with entries 0, 1, and 2, such that $\sum_{j=1}^{n} \epsilon_j = \sum_{j=1}^{n} \delta_j \leq 2m$ and $\sum_{j=1}^{n} \epsilon_j a_j = \sum_{j=1}^{n} \delta_j a_j$.

10738. *Proposed by Radu Theodorescu, Université Laval, Sainte-Foy, PQ, Canada.* For $t > 0$, let $m_n(t) = \sum_{k=0}^{\infty} k^n e^{-t} t^k / k!$ be the nth moment of a Poisson distribution with parameter t. Let $c_n(t) = m_n(t)/n!$. A sequence $a_0, a_1, \ldots$ is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n > 0$ and is *log-concave* if $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n > 0$.
(a) Show that $m_0(t), m_1(t), \ldots$ is log-convex.
(b) Show that $c_0(t), c_1(t), \ldots$ is not log-concave when $t < 1$.
(c) Show that $c_0(t), c_1(t), \ldots$ is log-concave when t is sufficiently large.
(d)* Is $c_0(t), c_1(t), \ldots$ log-concave when $t \geq 1$?

SOLUTIONS

Moments of Roots of Chebyshev Polynomials

10448 [1995, 360]. *Proposed by Fu-Chuen Chang, National Sun Yat-sen University, Kaohsiung, Taiwan.* Fix a positive integer n. Let $x_i = \cos\big((2i-1)\pi/(2n)\big)$ for $1 \leq i \leq n$, and let $c_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ for $k \in \mathbb{N}$. Show that

$$c_k = \begin{cases} 0 & \text{if } k = 1, 3, \ldots, 2n-1; \\ \binom{k}{k/2} 2^{-k} & \text{if } k = 0, 2, \ldots, 2n-2. \end{cases}$$

Solution I by Paul Deiermann, Louisiana State University, Shreveport, LA. When $k = 0$ and n is odd, the term for $j = (n+1)/2$ appears as 0^0, which must be taken to be 1 to arrive at the stated formula and our generalization. We show, for arbitrary integers $k \geq 0$, that

$$c_k = \begin{cases} 0 & \text{for } k \text{ odd}, \\ 2^{-k}\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}} & \text{for } k \text{ even}, \end{cases}$$

where $m = \lfloor k/(2n) \rfloor$. The stated problem covers those k for which $m = 0$.

First note that $x_{n+1-j} = -x_j$, so the terms of the sum cancel in pairs when k is odd. We may thus restrict to the case of k even. Since $x_j = \big(e^{i\pi(2j-1)/(2n)} + e^{-i\pi(2j-1)/(2n)}\big)/2$, the binomial theorem and a summation of a finite geometric progression imply

$$\sum_{j=1}^{n} x_j^k = \sum_{j=1}^{n} 2^{-k}\left(e^{i\pi\frac{2j-1}{2n}} + e^{-i\pi\frac{2j-1}{2n}}\right)^k = 2^{-k}\sum_{j=1}^{n}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(k-2q)} e^{i\frac{2\pi}{n}(q-k/2)j}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(k-2q)}\sum_{j=1}^{n} e^{i\frac{2\pi}{n}(q-k/2)j} = 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(2q-k)}\sum_{u=0}^{n-1} e^{i\frac{2\pi}{n}(q-k/2)u}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q} e^{i\frac{\pi}{2n}(2q-k)}\begin{cases} n & \text{if } q - k/2 = pn,\ p \in \mathbb{Z}, \\ \frac{1-e^{i\pi(2q-k)}}{1-e^{i\frac{2\pi}{n}(q-k/2)}} = 0 & \text{if } n \nmid q - k/2. \end{cases}$$

Since k is even, $q - k/2 = pn$ implies $q = pn + k/2$. Then, $0 \leq q \leq k$ gives $-m \leq p \leq m$. Also, in this case, $e^{i\frac{\pi}{2n}(2q-k)} = e^{i\pi p} = (-1)^p$. Thus, we get

$$\sum_{j=1}^{n} x_j^k = 2^{-k} n \sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}}.$$