Chaos, Cantor Sets, and Hyperbolicity for the Logistic Maps

Roger L. Kraft

The family of logistic maps $f_\mu(x) = \mu x(1-x)$ appears in almost every dynamical systems textbook. It is one of the simplest nonlinear systems that one can study, but it is amazingly rich in phenomena. It has a surprising number of connections to other topics in dynamical systems and applied mathematics, for example, population dynamics, symbolic dynamics, complex analytic dynamics, the Mandelbrot set, the period-doubling route to chaos, renormalization, universality, homoclinic bifurcations, horseshoes, and invariant measures. Because of its simplicity, many introductory dynamical systems textbooks use it as a primary example, in particular as the primary example of a chaotic dynamical system. When $\mu > 2 + \sqrt{5} \approx 4.236$, it is not too difficult to prove that f_μ is chaotic on an invariant Cantor set; for the details, see any of [**1**, pp. 31–50], [**3**, pp. 112–126], or [**4**, pp. 69–85]. Each of these books states without proof that f_μ is actually chaotic for all $\mu > 4$. Our goal is to give a simple proof of this fact.

As far as I know, only one textbook gives a proof that f_μ is chaotic for $\mu > 4$ [**6**, pp. 33–37]. However, its proof uses the Poincaré hyperbolic metric on the unit interval, the calculation of a derivative using different metrics, and the Schwarz Lemma from the theory of complex variables. While this proof is very elegant, and hints at the connections between the logistic maps and complex analytic dynamics, it is not in the spirit of the more elementary books.

The family of logistic maps $f_\mu : \mathbb{R} \to \mathbb{R}$, $\mu > 0$, is a family of parabolas that open downward, intercept the x-axis at 0 and 1, and have a maximum at $1/2$. Since the maximum value is $\mu/4$, f_μ maps the interval $[0,1]$ into $[0,1]$ when $0 < \mu \leq 4$. But when $\mu > 4$, there are points in $[0,1]$ that escape from $[0,1]$ under forward iteration of f_μ. Let

$$\Lambda_\mu \equiv \bigcap_{n=1}^{\infty} f_\mu^{-n}([0,1]).$$

For $\mu > 4$, Λ_μ contains exactly those points in $[0,1]$ that never escape under forward iteration by f_μ. Our main result is:

Theorem 1. *If $\mu > 4$, then Λ_μ is a Cantor set, and the restriction of f_μ to Λ_μ is chaotic.*

Once we have shown that Λ_μ is a Cantor set, the proof that the restriction of f_μ to Λ_μ is chaotic is same as in the case $\mu > 2 + \sqrt{5}$: Use itineraries to construct a topological conjugacy between f_μ on Λ_μ and the shift map σ on $\Sigma_2 = \{0,1\}^{\mathbb{N}}$; this shows that f_μ on Λ_μ is topologically transitive and has dense periodic points. It is easy to show that f_μ on Λ_μ has sensitive dependence on initial conditions; in fact, it is easy to show that it is expansive, which is a stronger property [**1**, p. 50], [**6**, p. 83]. All the details of these steps remain unchanged.

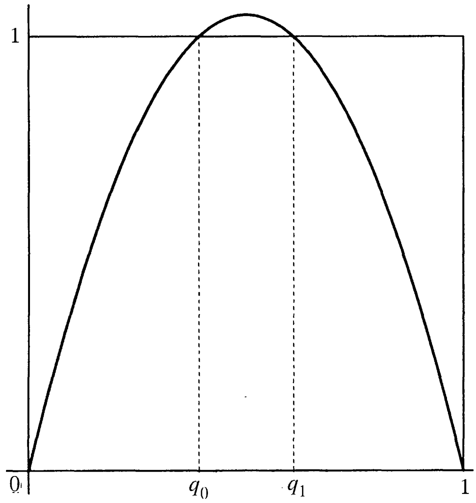

Figure 1.

Choose a value of $\mu > 4$, and let us define some notation. Let $q_0 < q_1$ solve $f_\mu(x) = 1$; see Figure 1. Let $I_0 = [0,1]$, and let $I_1 = [0, q_0] \cup [q_1, 1]$. Notice that $I_1 = f_\mu^{-1}(I_0)$. In general, let $I_n = f_\mu^{-1}(I_{n-1}) = f_\mu^{-n}(I_0)$. Then I_n is exactly those points in $[0,1]$ that stay in $[0,1]$ for their first n iterates under f_μ, and $I_n \subset I_{n-1}$. Notice also that I_n is made up of 2^n disjoint closed intervals. Order the 2^n components of I_n from left to right, and let $I_{n,j}$ denote the j-th component. (To keep the notation as simple as possible, we suppress the explicit dependence of objects like I_n on the parameter μ.) For more details about the definitions in this paragraph, see [**1**, pp. 34–36], [**3**, pp. 112–114], [**4**, pp. 70–73], or [**6**, pp. 30–32].

If I is an interval, we let $|I|$ denote the length of I.

Recall that a subset of the real line is a *Cantor set* if it is compact, perfect, and totally disconnected. Recall also that a subset of the real line is *totally disconnected* if and only if it does not contain any intervals; see [**1**, p. 37], [**3**, p. 116], [**4**, p. 73], or [**6**, p. 26].

The first step in proving that Λ_μ is a Cantor set is the following lemma.

Lemma 2. *If* $\mu > 4$, *then* Λ_μ *is a compact perfect set.*

Proof: Since $\Lambda_\mu = \cap_{n=0}^{\infty} I_n$ and each I_n is compact, we know that Λ_μ is compact. To show that Λ_μ is perfect, first notice that for every n, all the endpoints of I_n are contained in Λ_μ. Let $x \in \Lambda_\mu$, and for each n let I_{n,j_x} denote the component of I_n that contains x. If $|I_{n,j_x}| \to 0$ as $n \to \infty$, then there are endpoints from I_{n,j_x} arbitrarily close to x, so x is in the closure of $\Lambda_\mu \setminus \{x\}$. On the other hand, if $|I_{n,j_x}|$ does not go to 0 as $n \to \infty$, then $\cap_{n=0}^{\infty} I_{n,j_x}$ is a closed interval, and $x \in \cap_{n=0}^{\infty} I_{n,j_x} \subset \Lambda_\mu$, so once again x is in the closure of $\Lambda_\mu \setminus \{x\}$. This shows that Λ_μ is perfect. ∎

To finish the proof that Λ_μ is a Cantor set, we need to show that it does not contain any intervals. How is this done when $\mu > 2 + \sqrt{5}$? A simple calculation shows that when $\mu = 2 + \sqrt{5}$, $f_\mu'(q_0) = 1$. So if $\mu > 2 + \sqrt{5}$, then $|f_\mu'(x)| > 1$ for all $x \in I_1$. This key fact makes the case $\mu > 2 + \sqrt{5}$ straightforward, as we now show.

Lemma 3. *If $\mu > 2 + \sqrt{5}$, then Λ_μ is a Cantor set.*

Proof: Suppose that Λ_μ contains an interval; let $[a, b] \subset \Lambda_\mu$. For every $n \geq 1$, the Mean Value Theorem applied to f_μ^n on the interval $[a, b]$ ensures that there is a point $c_n \in (a, b)$ such that

$$f_\mu^n(b) - f_\mu^n(a) = \left(f_\mu^n\right)'(c_n)(b-a).$$

Let $\lambda = f_\mu'(q_0)$, so $|f_\mu'(x)| \geq \lambda$ for all $x \in I_1$. Since $\mu > 2 + \sqrt{5}$, we have $\lambda > 1$. Since $c_n \in [a, b] \subset \Lambda_\mu$, we have $f_\mu^i(c_n) \in \Lambda_\mu \subset I_1$ for all $0 \leq \mathrm{i} \leq \mathrm{n} - 1$. Therefore, $|(f_\mu^n)'(c_n)| \geq \lambda^n$ by the chain rule, and

$$\left|f_\mu^n(b) - f_\mu^n(a)\right| = \left|\left(f_\mu^n\right)'(c_n)\right| \cdot |b-a| \geq \lambda^n |b-a|.$$

Since $\lambda > 1$, this implies that $|f_\mu^n(b) - f_\mu^n(a)| > 1$ for all n sufficiently large. But Λ_μ is invariant, and hence $\{f^n(a), f^n(b)\} \subset \Lambda_\mu \subset [0, 1]$ for all n, so we have a contradiction. Thus Λ_μ does not contain any intervals, and hence is a Cantor set. ∎

Here is another way to think about this proof: If $\mu > 2 + \sqrt{5}$, and we apply f_μ^{-1} to I_n to get I_{n+1}, then f_μ^{-1} shrinks the length of every component of I_n by at least the amount $\lambda^{-1} < 1$, so the lengths of the components of I_n go to zero as n goes to infinity.

When $\mu \in (4, 2 + \sqrt{5}]$, we have $|f_\mu'(x)| > 1$ for some $x \in I_1$, but $|f_\mu'(x)| \leq 1$ for other $x \in I_1$. When we apply f_μ^{-1} to I_n to get I_{n+1}, f_μ^{-1} shrinks some components of I_n, but, in contrast to the case when $\mu > 2 + \sqrt{5}$, f_μ^{-1} may also stretch other components of I_n. This combination of shrinking and stretching by f_μ^{-1} is what makes it difficult to show that Λ_μ is a Cantor set when $4 < \mu \leq 2 + \sqrt{5}$. However, a little playing around with f_μ should give one the sense that somehow, the stretching is eventually dominated by the shrinking as we repeatedly apply f_μ^{-1}. This leads to the following important definition.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function, and suppose that Λ is a compact invariant set for f (i.e., $f(\Lambda) = \Lambda$). Then Λ is a *hyperbolic set* for f if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and all $n \geq 1$.

The C in the definition takes care of the fact that f^{-1} may stretch some intervals (i.e., $|f'(x)| \leq 1$ for some $x \in \Lambda$), in which case $C < 1$, but $\lambda > 1$ implies that shrinking under f^{-n} eventually dominates any stretching when $C\lambda^n > 1$; see [**6**, pp. 107–108 and p. 156].

The following lemma gives some insight into the definition of hyperbolicity, and makes it easier to use.

Lemma 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function, and suppose that Λ is a compact invariant set for f. Then the following are equivalent.*

(1) *There are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and all $n \geq 1$.*
(2) *There is an integer $N \geq 1$ such that $|(f^n)'(x)| > 1$ for all $x \in \Lambda$ and all $n \geq N$.*
(3) *There is an integer $n_0 \geq 1$ such that $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$.*
(4) *For every $x \in \Lambda$ there is an integer $n_x \geq 1$, which may depend on x, such that $|(f^{n_x})'(x)| > 1$.*

Remark. If $|f'(x)| > 1$ for all $x \in \Lambda$, then it is obvious that all four of the conditions in the lemma are true. This emphasizes once again that it is the possibility that $|f'(x)| \le 1$ for some $x \in \Lambda$ that makes the definition of hyperbolicity subtle.

Proof: (4) $\Rightarrow$ (3) [**5**, p. 220] Since f is C^1, $(f^n)'$ is continuous for every n. For each $x \in \Lambda$, $|(f^{n_x})'(x)| > 1$ and $(f^{n_x})'$ is continuous, so there is neighborhood U_x of x and a $\lambda_x > 1$ such that $|(f^{n_x})'(y)| > \lambda_x$ for all $y \in U_x$. The open sets $\{U_x | x \in \Lambda\}$ cover the compact set Λ, so there is a finite subcover $\{U_i\}_{i=1}^k$, numbers $\{\lambda_i\}_{i=1}^k$ all strictly greater than 1, and integers $\{n_i\}_{i=1}^k$ such that $|(f^{n_i})'(y)| > \lambda_i$ for all $y \in U_i$. Let

$$\nu = \max\{n_1 \dots n_k\}, \quad \lambda_0 = \min\{\lambda_1 \dots \lambda_k\}, \quad \text{and} \quad m = \min_{x \in \Lambda}\{|f'(x)|\},$$

so $m > 0$ (why?). Choose an integer k so that $\lambda_0^k m^\nu > 1$, and let $n_0 = k\nu + \nu$. Now that we have defined our global choice for n_0, we need to show that $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$. If we imagine that λ_0 represents "good" derivatives ($\lambda_0 > 1$) and m represents "bad" derivatives ($m < 1$), then we need to show that $f^{n_0}(x)$ contains at least k iterates with good derivatives to compensate for the worst case of ν iterates with bad derivatives.

Choose $x \in \Lambda$ and perform the following selection process that depends on x and terminates after a finite number of steps:

Choose ν_1 so that $x \in U_{\nu_1}$. Now suppose that we are given $\{\nu_1, \dots, \nu_j\}$. Let $\eta = \sum_{i=1}^j n_{\nu_i}$. Choose ν_{j+1} so that $f^\eta(x) \in U_{\nu_{j+1}}$. If $\eta + \nu_{j+1} > k\nu$, then stop; otherwise, go on to choose ν_{j+2}.

If the selection process stops after j steps, then $k\nu < \sum_{i=1}^j n_{\nu_i} \le k\nu + \nu$. Write $n_0 = n_{\nu_1} + n_{\nu_2} + \cdots + n_{\nu_j} + i_x$, where $0 \le i_x \le \nu$. Each n_{ν_i} represents a good iterate (derivative > 1), j represents how many good iterates we actually have, and i_x represents how many bad iterates (derivatives < 1) we actually have. Since each $n_{\nu_i} \le \nu$, we know that $j \ge k$. Using the chain rule, we can estimate $|(f^{n_0})'(x)|$:

$$\begin{aligned}
|(f^{n_0})'(x)| &= \left|\left(f^{i_x}(f^{n_{\nu_j}}(f^{n_{\nu_{j-1}}} \dots f^{n_{\nu_1}}))\right)'(x)\right| \\
&\ge m^{i_x}\lambda_{\nu_j}\lambda_{\nu_{j-1}} \dots \lambda_{\nu_1} && \left(\text{by the properties of the subcover } \{U_i\}_{i=1}^k\right) \\
&\ge m^\nu\lambda_{\nu_j}\lambda_{\nu_{j-1}} \dots \lambda_{\nu_1} && (\text{since } m \le 1 \text{ and } i_x \le \nu) \\
&\ge m^\nu\lambda_0^j && (\text{by our choice of } \lambda_0) \\
&\ge m^\nu\lambda_0^k && (\text{since } j \ge k \text{ and } \lambda_0 > 1) \\
&> 1 && (\text{because of our choice of } k).
\end{aligned}$$

Thus, $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$, which proves (3).

(3) $\Rightarrow$ (2) [**1**, p. 99] If $n_0 = 1$ in (3), there is nothing to prove, so suppose $n_0 > 1$. Let

$$\lambda = \min_{x \in \Lambda}\{|(f^{n_0})'(x)|\} \quad \text{and} \quad m = \min_{x \in \Lambda}\{|f'(x)|\},$$

so $\lambda > 1$ (why?) and $m > 0$. Since we are assuming that $n_0 > 1$, we must have $m \le 1$. Choose k so that $m^{n_0 - 1}\lambda^k > 1$. Let $N = n_0 k + (n_0 - 1)$. If $n > N$, write

$n = n_0(k + \nu) + i$, where $\nu > 0$ and $0 \le i \le n_0 - 1$. Then for any $x \in \Lambda$ we have

$$\begin{aligned}
|(f^n)'(x)| &= |(f^{n_0(k+\nu)})'(f^i(x))| \cdot |(f^i)'(x)| \\
&\ge \lambda^{k+\nu} m^i \\
&\ge \lambda^{\nu}\lambda^k m^{n_0 - 1} && (\text{since } m \le 1 \text{ and } i \le n_0 - 1) \\
&> \lambda^{\nu} && (\text{by our choice of } k) \\
&> 1 && (\text{since } \lambda > 1).
\end{aligned}$$

(2) $\Rightarrow$ (1) If $N = 1$ in (2), there is nothing to prove, so suppose $N > 1$. Let

$$m_1 = \min_{x \in \Lambda}\{|(f^N)'(x)|\} \quad \text{and} \quad m = \min_{x\in\Lambda}\{|f'(x)|\},$$

so $m_1 > 1$. Since we are assuming that $N > 1$, we must have $m \le 1$. Let

$$\lambda = m_1^{1/N} \quad \text{and} \quad C = (m/\lambda)^{N-1},$$

so $\lambda > 1$ and $C > 0$. For any $n > 0$ write $n = kN + i$, where $k \ge 0$ and $0 \le i \le N - 1$. Then for any $x \in \Lambda$ we have

$$\begin{aligned}
|(f^n)'(x)| &= |(f^{kN})'(f^i(x))| \cdot |(f^i)'(x)| \\
&\ge m_1^k m^i \\
&= \lambda^{kN} m^i && (\text{by our choice of } \lambda) \\
&= \lambda^{kN}\lambda^i (m/\lambda)^i \\
&\ge \lambda^{kN+i}(m/\lambda)^{N-1} && (\text{since } m/\lambda < 1 \text{ and } i \le N - 1) \\
&= C\lambda^n && (\text{by our choice of } C).
\end{aligned}$$

(1) $\Rightarrow$ (4) Choose n large enough so that $C\lambda^n > 1$. Then we have $|(f^n)'(x)| \ge C\lambda^n > 1$. Now let $n_x = n$ for every $x \in \Lambda$. ∎

Why so many versions of the definition of hyperbolic? When we want to prove that a set is hyperbolic, it helps to use the weakest version of the definition, (4). On the other hand, when we want to prove general conclusions about a hyperbolic set, then it helps to use the strongest version, (1). Also, (2) is used as the definition of a hyperbolic set in some textbooks when the emphasis is on dynamics in one dimension, e.g., [**1**, p. 38], or [**4**, p. 77]. But a generalization of (1) is used in the definition of hyperbolicity for higher dimensions, e.g., [**6**, p. 241].

When $\mu > 2 + \sqrt{5}$, we have $|f_\mu'(x)| > 1$ for all $x \in I_1$, and this is the key to proving that Λ_μ is a Cantor set. To prove that Λ_μ is a Cantor set when $\mu > 4$, we need to replace "$|f_\mu'(x)| > 1$ for all $x \in I_1$" with "Λ_μ is a hyperbolic set for f_μ." Before we can begin the proof of hyperbolicity, we need to introduce an important tool, the Schwarzian derivative.

Definition. The *Schwarzian derivative* of a C^3 function f at a point x where $f'(x) \ne 0$ is

$$Sf(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

This strange definition turns out to be tremendously useful. Our first result about the Schwarzian derivative is the following lemma.

Lemma 5. *For the logistic family f_μ with $\mu > 0$,*

(1) $Sf_\mu(x) < 0$ *for all* $x \in \mathbb{R} \setminus \{1/2\}$,
(2) $Sf_\mu^n(x) < 0$ *for all* $n > 1$ *and all* $x \in \mathbb{R} \setminus \cup_{i=0}^{n-1} f_\mu^{-i}(1/2)$.

The first item is easy, since $f_\mu''' \equiv 0$. However, the second is not so obvious, since f_μ^n is a polynomial of degree 2^n. The second item follows from the first item, the following lemma, and induction. This "hereditary" result is one of the reasons the Schwarzian derivative is so useful.

Lemma 6. *If $g'(x) \neq 0$ and $f'(g(x)) \neq 0$, then*

$$S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x).$$

So if $Sg(x) < 0$ and $Sf(g(x)) < 0$, then $S(f \circ g)(x) < 0$.

Proof: The chain rule gives

$$(f \circ g)'(x) = f'(g(x))g'(x),$$

$$(f \circ g)''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x), \quad \text{and}$$

$$(f \circ g)'''(x) = f'''(g(x))(g'(x))^3 + 3f''(g(x))g''(x)g'(x) + f'(g(x))g'''(x).$$

A computation now gives the desired result. ■

We say that a function f has *negative Schwarzian derivative on an interval I* if $f'(x) \neq 0$ and $Sf(x) < 0$ for all $x \in I$; we abbreviate this as $Sf < 0$ on I. The following lemma gives a geometric consequence of negative Schwarzian derivative.

Lemma 7. *If I is an open interval and $Sf < 0$ on I, then f' cannot have a positive local minimum on I, nor can it have a negative local maximum.*

Proof: Suppose that x is a positive local minimum point for f' on I. Then $f'(x) > 0$, $f''(x) = 0$, and $f'''(x) \geq 0$ (why?). This implies that $Sf(x) \geq 0$, which contradicts $Sf < 0$ on I.

Similarly, if x is a negative local maximum point for f' on I, then $f'(x) < 0$, $f''(x) = 0$, and $f'''(x) \leq 0$. This implies $Sf(x) \geq 0$, which contradicts $Sf < 0$ on I. ■

Lemma 8 (Minimum Principle). *Let $I = [a, b]$ and suppose f is C^3 on I. If $Sf < 0$ on (a, b), then $|f'(x)| > \min\{|f'(a)|, |f'(b)|\}$ for all $x \in (a, b)$.*

Proof: Since $|f'|$ is continuous on the closed interval I, it must have a minimum at some point $x_0 \in I$. If $x_0 \in (a, b)$, then $f'(x_0) \neq 0$ since $Sf < 0$ on (a, b). If $f'(x_0) > 0$, then f' has a positive local minimum on (a, b), which contradicts Lemma 7. On the other hand, if $f'(x_0) < 0$, then f' has a negative local maximum on (a, b), another contradiction of Lemma 7. It follows that $x_0 = a$ or $x_0 = b$. ■

The Minimum Principle is the key result we need about negative Schwarzian derivatives. When we apply it to the iterates f_μ^n, we get information about the shape of the graph of f_μ^n between its critical points that would be very difficult to get in any other way.

There are other important consequences for one-dimensional dynamical systems of negative Schwarzian derivative; see [**1**, Section 1.11].

Now consider f_μ. For any $\mu > 0$, f_μ has a fixed point at $p_1 = 1 - (1/\mu)$, and $f'_\mu(p_1) = 2 - \mu$, so $|f'_\mu(p_1)| > 1$ when $\mu > 3$. Let $p_0 = 1/\mu$, so p_0 and p_1 are symmetric about $1/2$; see Figure 2. Notice that $f_\mu(p_0) = p_1$ (when $\mu > 2$), and that $f_\mu([p_0, q_0]) = f_\mu([q_1, p_1]) = [p_1, 1]$. So if we let $J = (p_0, q_0) \cup (q_1, p_1)$, and if x is any point in J, then $f_\mu(x) \notin J$. But we have the following "return lemma."

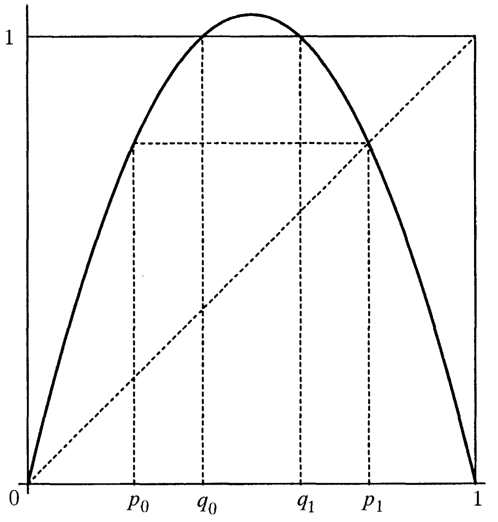

Figure 2.

Lemma 9 (Return Lemma). *If $\mu > 4$ and if $x \in J$, then there is an integer $n \geq 2$ such that $f_\mu^n(x) \in [p_0, p_1)$.*

Proof: Choose $x \in J$, so $f_\mu(x) \in (p_1, 1)$ and $f_\mu^2(x) \in (0, p_1)$. If $f_\mu^2(x) \in [p_0, p_1)$, then we are done. Suppose that $f_\mu^2(x) \in (0, p_0)$. We claim that for some $n \geq 1$, $f_\mu^{2+n}(x)$ is in $[p_0, p_1)$. Suppose not. Since $f_\mu(z) > z$ for all $z \in (0, p_0)$, we know that $f_\mu^{2+n}(x)$ is an increasing sequence bounded from above by p_0. So $f_\mu^{2+n}(x)$ converges as $n \to \infty$ to some point $z_0 \leq p_0$. It follows that z_0 is a fixed point for f_μ. But $0 < z_0 < p_1$, so we have a contradiction. ∎

Lemma 10. *If $\mu > 4$, then $q_0 - p_0 < p_0$. So the intervals (p_0, q_0) and (q_1, p_1) are shorter than the intervals $(0, p_0)$ and $(p_1, 1)$.*

Proof: We need to show that $\mu > 4$ implies $2p_0 > q_0$. Recall that

$$p_0 = \frac{1}{\mu} \quad \text{and} \quad q_0 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}.$$

Since $\mu > 4$, we have $0 < 1 - (4/\mu) < 1$, so $\sqrt{1-(4/\mu)} > 1 - (4/\mu)$. After multiplying both sides by $1/2$, we have

$$\sqrt{\frac{1}{4} - \frac{1}{\mu}} > \frac{1}{2} - \frac{2}{\mu},$$

or

$$2\left(\frac{1}{\mu}\right) > \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}. \qquad \blacksquare$$

Now we have all the ingredients we need to prove hyperbolicity.

Theorem 11. *If $\mu > 4$, then Λ_μ is a hyperbolic set for f_μ.*

Proof: Let $x \in \Lambda_\mu$. Assume that $x > 1/2$; the case where $x < 1/2$ follows by the symmetry of f_μ about $1/2$. We need to find an integer n (which may depend on x) such that $|(f_\mu^n)'(x)| > 1$. The hyperbolicity of Λ_μ then follows from Lemma 4.

If $x \geq p_1$, we can let $n = 1$ (why?). If $x = q_1$, then $f_\mu^n(q_1) = 0$ for $n \geq 2$, so

$$\left|(f_\mu^n)'(q_1)\right| = \left|f_\mu'(q_1)\right| \cdot \left|f_\mu'(1)\right| \cdot \left|f_\mu'(0)\right|^{n-2} = \mu^{n-1}\sqrt{\mu^2 - 4\mu} = \mu^n\sqrt{1-(4/\mu)},$$

which is strictly greater than 1 for all n sufficiently large.

Now concentrate on x between q_1 and p_1. The Return Lemma ensures that there is an n such that $f_\mu^n(x) \in [p_0, p_1)$. Let $I_{n,j}$ be the component of I_n that contains x. There are two cases to consider: either $I_{n,j} \subset [q_1, p_1)$, or it is not.

Suppose $I_{n,j} \subset [q_1, p_1)$. Since f_μ^n maps $I_{n,j}$ monotonically onto $[0,1]$ (see [**1**, p. 36], [**4**, p. 71]), or [**6**, p. 31]), we can partition $I_{n,j}$ into three subintervals, $I_{n,j} = L_{n,j} \cup K_{n,j} \cup R_{n,j}$, where $f_\mu^n(L_{n,j}) = [0, p_0]$, $f_\mu^n(K_{n,j}) = (p_0, p_1)$, and $f_\mu^n(R_{n,j}) = [p_1, 1]$. Since $L_{n,j} \subset I_{n,j} \subset [q_1, p_1)$ and $R_{n,j} \subset I_{n,j} \subset [q_1, p_1)$, Lemma 10 ensures that $|f_\mu^n(L_{n,j})| > |L_{n,j}|$ and $|f_\mu^n(R_{n,j})| > |R_{n,j}|$. That is, f_μ^n must do some stretching near both ends of $I_{n,j}$. By the Mean Value Theorem applied to f_μ^n, there is a point $y \in L_{n,j}$ and a point $z \in R_{n,j}$ such that $|(f_\mu^n)'(y)| > 1$ and $|(f_\mu^n)'(z)| > 1$. Since $f_\mu^n(x) \in [p_0, p_1)$, we have $x \in \text{closure}(K_{n,j})$, so $y \leq x < z$. Since f_μ^n does not have a critical point in $[y, z]$, the Minimum Principle ensures that $|(f_\mu^n)'(x)| > 1$.

Now suppose that $I_{n,j}$ is not a subset of $[q_1, p_1)$. Once again, partition $I_{n,j}$ into three subintervals, $I_{n,j} = L_{n,j} \cup K_{n,j} \cup R_{n,j}$, where $f_\mu^n(L_{n,j}) = [0, p_0]$, $f_\mu^n(K_{n,j}) = (p_0, p_1)$, and $f_\mu^n(R_{n,j}) = [p_1, 1]$. As before, $x \in \text{closure}(K_{n,j})$ because $f_\mu^n(x) \in [p_0, p_1)$. Since $x \in (q_1, p_1)$, one of $L_{n,j}$ or $R_{n,j}$ is contained in $[q_1, p_1)$, but, since $I_{n,j}$ is not a subset of $[q_1, p_1)$, the other one of $L_{n,j}$ or $R_{n,j}$ is not contained in $[q_1, p_1)$. Suppose that $L_{n,j}$ is contained in $[q_1, p_1)$, and $R_{n,j}$ is not (the other case is similar). Since $I_{n,j} \subset [q_1, 1]$ and $I_{n,j} \cap [q_1, p_1) \neq \varnothing$, it must be that $p_1 \in I_{n,j}$. As before, $|f_\mu^n(L_{n,j})| > |L_{n,j}|$, so the Mean Value Theorem ensures that there is a point $y \in L_{n,j}$ such that $|(f_\mu^n)'(y)| > 1$. And $|(f_\mu^n)'(p_1)| > 1$ since p_1 is a hyperbolic repelling fixed point. Then $x \in [y, p_1]$ and f_μ^n does not have a critical point in $[y, p_1]$, so the Minimum Principle ensures that $|(f_\mu^n)'(x)| > 1$. $\blacksquare$

Remark. This proof of the hyperbolicity of Λ_μ is adapted from the idea of an "induced map" or "first return map" for f_μ; see [**2**, p. 341], and see [**1**, pp. 75–78] for an application of this idea when $\mu = 4$.

Theorem 12. *If $\mu > 4$, then Λ_μ is a Cantor set.*

Proof: The proof is now as easy as the case $\mu > 2 + \sqrt{5}$. Just observe that since $c_n \in \Lambda_\mu$, the hyperbolicity of Λ_μ ensures that $|(f_\mu^n)'(c_n)| \geq C\lambda^n$. The rest of the proof is unchanged. ■

REFERENCES

1. R. Devaney, *An introduction to chaotic dynamical systems*, 2nd ed., Addison-Wesley, Redwood City, CA, 1989.
2. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences, 42, Springer-Verlag, New York-Berlin, 1983.
3. D. Gulick, *Encounters with chaos*, McGraw-Hill, New York, 1992.
4. R. Holmgren, *A First Course in Discrete Dynamical Systems*, 2nd ed., Springer-Verlag, New York, 1996.
5. W. de Melo and S. van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1993.
6. C. Robinson, *Stability, symbolic dynamics, and chaos*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

ROGER KRAFT received a Ph.D. from Northwestern University in 1990 under Clark Robinson. He was a visiting assistant professor at the University of Cincinnati, Case Western Reserve University, and Pomona College, and a postdoc at the M.S.R.I. He is now an associate professor at Purdue University Calumet. His field of research is dynamical systems, with special emphasis on two chaotic little attractors named Peter and Ziyad.

Department of Mathematics, Computer Science, and Statistics, Purdue University Calumet, Hammond, IN 46323

roger@calumet.purdue.edu

From the MONTHLY 100 Years Ago

The following are some of the advanced courses of Mathematics offered for the year 1899–1900 at the University of Chicago: Twisted Curves and Surfaces, Associate Professor Maschke; Projective Geometry, Professor Moore; Theory of Invariants, Professor Bolza; Continuous Groups, Professor Bolza; Theory of Functions of a Complex Variable, Professor Moore and Associate Professor Maschke; Elliptic Functions, Professor Bolza; Hyperelliptic Functions, Professor Bolza; Abstract Groups, Associate Professor Maschke; Elliptic Modular Functions, Professor Moore; Theory of Substitution, Professor Moore; Theory of Numbers, Assistant Professor Young, etc., etc.

MONTHLY 6 (1899) 158