Solution by the editors. Let $f_k(s) = \sum_{n=1}^{k} z_n s^n$ and $f(s) = \sum_{n=1}^{\infty} z_n s^n$. Since z_{n+1} is a convex combination of $\{z_1, \ldots, z_m\}$, the sequence is bounded and indeed $|z_n| \le \max\{|z_1|, |z_2|, \ldots, |z_m|\}$, so the radius of convergence for $f(s)$ is positive. From the recurrence relation we have

$$f(s) = f_m(s) + e^{i\omega} \sum_{k=1}^{m} p_k s^k \big(f(s) - f_{m-k}(s)\big).$$

Thus $f(s)$ is a rational function of s, say $f(s) = A(s)/B(s)$.

Assume first that $e^{i\omega} = 1$. Now $B(s) = 1 - \sum_{k=1}^{m} p_k s^k$ has a zero at $s = 1$ since $\sum_{k=1}^{m} p_k = 1$. It is a simple zero since $B'(1) = -\sum_{k=1}^{m} k p_k$ is not zero. But $B(s)$ has no other zeros on or inside the unit disk, since if $|s| \le 1$, then $|\sum_{k=1}^{m} p_k s^k| \le \sum_{k=1}^{m} p_k |s^k| \le \sum_{k=1}^{m} p_k = 1$, with equality only if $|s| = 1$ and all s^k have the same argument. Thus we have a partial fraction expansion $f(s) = A(1)/(B'(1)(s-1)) + C(s)$ where $C(s)$ is a rational function of s with all poles outside the unit disk. The Maclaurin series for $C(s)$ has radius of convergence greater than 1, so its coefficients go to zero, while the Maclaurin series for $A(1)/(B'(1)(s-1))$ has all coefficients equal to $-A(1)/B'(1)$. It follows that

$$\lim_{n\to\infty} z_n = -\frac{A(1)}{B'(1)} = \frac{\sum_{k=1}^{m} p_k \sum_{n=m-k+1}^{m} z_n}{\sum_{k=1}^{m} k p_k}.$$

If $e^{i\omega} \ne 1$, then $B(s)$ has all its roots outside the closed unit disk, so we see that $z_n \to 0$ without a partial fraction argument.

Solved also by D. M. Bradley, B. Burdick, D. Callan, R. J. Chapman (U. K.), G. Keselman, J. H. Lindsey II, V. Lucic (Canada), W. A. Newcomb, P. Szeptycki, E. I. Verriest, and the proposers.

Random Polynomials with Real Roots

10660 [1998, 366]. *Proposed by Colin L. Mallows, AT&T Laboratories, Florham Park, NJ.* Suppose the coefficients of a polynomial are independent Gaussian random variables, each with mean 0. For each $\epsilon > 0$, can the variances be chosen so that all of the zeroes of the polynomial are real with probability at least $1 - \epsilon$?

Solution by Kenneth Schilling, University of Michigan–Flint, Flint, MI. We prove the following slightly stronger claim by induction on n.

Proposition. *Fix $\varepsilon > 0$ and $n \ge 1$. There exist $\sigma_0, \ldots, \sigma_n > 0$ such that, if (a) $a_0, \ldots, a_n$ are independent Gaussian random variables with mean 0, (b) each a_i has variance σ_i^2, and (c) S is the event $\{a_0 + a_1 x + \cdots + a_n x^n = 0$ has n distinct nonzero real solutions $x\}$, then S has probability at least $1 - \varepsilon$.*

Proof. This is obvious for $n = 1$. To complete the induction, let n and $\varepsilon > 0$ be given. Let $\sigma_0^2, \ldots, \sigma_n^2$ be variances as provided by the induction hypothesis applied to $\varepsilon/2$ and n. Let $f(x)$ denote the random polynomial $a_0 + a_1 x + \cdots + a_n x^n$, and let $g(x) = x f(x)$. Then on the event S, the function g has $n+1$ distinct real zeros, the derivative g' has n real zeros $y_1 < \cdots < y_n$, and the numbers $g(y_i)$ are nonzero and alternate in sign. Hence if $|\delta| < \min\{|g(y_1)|, \ldots, |g(y_n)|\}$, then $h(x) = g(x) + \delta$ has $n+1$ distinct real zeros.

Define the random variable $M = \min\{|g(x)| : x \in \mathbb{R}, g'(x) = 0\}$. Since $M > 0$ on S, there exists $\delta > 0$ such that the probability of the event $S \cap \{M \ge \delta\}$ is at least $1 - \varepsilon/2$. Now let b be a Gaussian random variable, independent of $a_0, \ldots, a_n$, with mean 0 and variance σ^2, where σ^2 is chosen so that $|b| < \delta$ with probability at least $1 - \varepsilon/2$. Then the event $S \cap \{M \ge \delta\} \cap \{0 < |b| < \delta\}$ has probability at least $(1 - \varepsilon/2)^2 > 1 - \varepsilon$, and on this event the equation $h(x) = b + x f(x) = b + a_0 x + a_1 x^2 + \cdots + a_n x^{n+1} = 0$ has $n+1$ distinct nonzero real solutions.

Solved also by J. H. Lindsey II, GCHQ Problems Group (U. K.), and the proposer.