An Elementary View of Euler's Summation Formula

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1. INTRODUCTION. The integral test for convergence of infinite series compares a finite sum $\sum_{k=1}^{n} f(k)$ and an integral $\int_1^n f(x)\,dx$, where f is positive and strictly decreasing. The difference between a sum and an integral can be represented geometrically, as indicated in Figure 1. In 1736, Euler [**3**] used a diagram like this to obtain the simplest case of what came to be known as Euler's summation formula, a powerful tool for estimating sums by integrals, and also for evaluating integrals in terms of sums. Later Euler [**4**] derived a more general version by an analytic method that is very clearly described in [**5**, pp. 159–161]. Colin Maclaurin [**9**] discovered the formula independently and used it in his *Treatise of Fluxions*, published in 1742, and some authors refer to the result as the Euler-Maclaurin summation formula. The general formula (24) is widely used in numerical analysis, analytic number theory, and the theory of asymptotic expansions. It contains Bernoulli numbers and periodic Bernoulli functions and is ordinarily discussed in courses in advanced calculus or real and complex analysis. This note shows how the general formula can be discovered by an elementary method, beginning with the diagram in Figure 1. This approach also shows how Bernoulli numbers and Bernoulli functions arise naturally along the way. The author has used this treatment successfully with beginning calculus students acquainted with the integral test.

2. GENERALIZED EULER'S CONSTANT. Throughout this section we assume that f is a positive and strictly decreasing function on $[1, \infty)$. We introduce a sequence $\{d_n\}$ of numbers that represent the sum of the areas of the shaded curvilinear pieces above the interval $[1, n]$ in Figure 1. That is, we define

$$d_n = \sum_{k=1}^{n-1} f(k) - \int_1^n f(x)\,dx, \qquad n = 2, 3, \ldots . \tag{1}$$

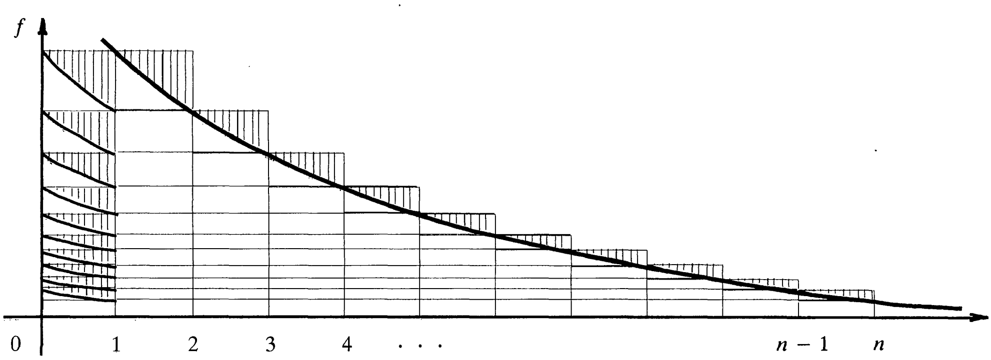

Figure 1. All the shaded regions above $[1, n]$ fit inside a rectangle of area $f(1)$.

It is clear that $d_{n+1} > d_n$ and that all the shaded pieces can be translated to the left to occupy a portion of the rectangle of altitude $f(1)$ above the interval $[0,1]$, as shown in Figure 1. Because f is decreasing there is no overlapping of the translated shaded pieces. Comparison of areas gives us the inequalities $0 < d_n < d_{n+1} < f(1)$. Therefore $\{d_n\}$ is increasing and bounded above, so it has a finite limit $C(f) = \lim_{n\to\infty} d(n)$. We refer to $C(f)$ as the *generalized Euler's constant* associated with the function f. Geometrically, $C(f)$ represents the sum of the areas of *all* the curvilinear triangular pieces over the interval $[1,\infty)$. These pieces can be translated to fit inside the rectangle of area $f(1)$ shown in Figure 1 (without overlapping), so we have the inequalities $0 < C(f) < f(1)$. Moreover, $C(f) - d_n$ represents the sum of the areas of the triangular pieces over the interval $[n,\infty)$. These pieces can be translated to the left to occupy (without overlapping) a portion of the rectangle of height $f(n)$ above the interval $[n, n+1]$. Comparing areas we find

$$0 < C(f) - d_n < f(n), \qquad n = 2,3,\dots. \tag{2}$$

From these inequalities we can easily deduce:

Theorem 1. *If f is positive and strictly decreasing on* $[1,\infty)$ *there is a positive constant* $C(f) < f(1)$ *and a sequence* $\{E_f(n)\}$, *with* $0 < E_f(n) < f(n)$, *such that*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n), \quad n = 2,3,\dots. \tag{3}$$

Note. Eq. (3) tells us that the difference between the sum and the integral is equal to a constant (depending on f) plus a positive quantity $E_f(n)$ smaller than the last term in the sum. Hence, if $f(n)$ tends to 0 as $n \to \infty$, then $E_f(n)$ also tends to 0.

Proof: If we define $E_f(n) = f(n) + d_n - C(f)$, then (3) follows from the definition (1), and the inequality $0 < E_f(n) < f(n)$ follows from (2). ∎

If $f(n) \to 0$ as $n \to \infty$, then (3) implies

$$C(f) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right). \tag{4}$$

Example. When $f(x) = 1/x$, $C(f)$ is the classical *Euler's constant*, often denoted by C (or by γ), and (4) states that $C = \lim_{n\to\infty}(\sum_{k=1}^{n}(1/k) - \log n)$. It is not known (to date) whether Euler's constant is rational or irrational. Its numerical value, correct to 20 decimals, is $C = 0.57721566490153286060$. In this case, Theorem 1 says that

$$\sum_{k=1}^{n}\frac{1}{k} = \log n + C + E(n), \quad \text{where } 0 < E(n) < \frac{1}{n}.$$

3. VARIOUS FORMS OF EULER'S SUMMATION FORMULA. In this section we no longer assume that f is positive or decreasing. At the outset we require only that the integral $\int_1^n f(x)\,dx$ exists for each integer $n \ge 2$. The key insight is to notice that the difference d_n in (1) can be written as

$$d_n = \sum_{k=1}^{n-1} I(k), \tag{5}$$

where

$$I(k) = \int_k^{k+1} \{f(k) - f(x)\}\,dx. \tag{6}$$

When f is positive and decreasing, as in Figure 2, $I(k)$ is the area of the shaded curvilinear triangular piece over the interval $[k, k+1]$. However, (5) and (6) are meaningful for any integrable f.

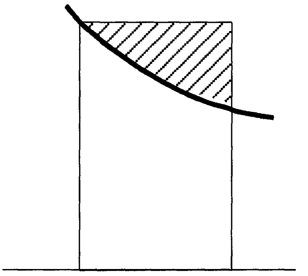

Figure 2. Geometric interpretation of the integral $I(k)$ as the area of the shaded region.

The integrand in (6) has the form $u\,dv$, where $u = f(k) - f(x)$ and $v = x + c$, where c is any constant. If we choose $c = -(k+1)$ and integrate by parts (assuming that f has a continuous derivative), the integrated part vanishes and the integral $I(k)$ reduces to

$$I(k) = \int_k^{k+1} (x - k - 1) f'(x)\,dx.$$

In this integral the dummy symbol x varies from k to $k+1$, so the quantity k in the integrand can be replaced by $[x]$, the greatest integer $\leq x$. Make this replacement and substitute in (6) to find

$$\begin{aligned} d_n = \sum_{k=1}^{n-1} I(k) &= \sum_{k=1}^{n-1} \int_k^{k+1} (x - [x] - 1) f'(x)\,dx \\ &= \int_1^n (x - [x]) f'(x)\,dx - \int_1^n f'(x)\,dx \\ &= \int_1^n (x - [x]) f'(x)\,dx + f(1) - f(n). \end{aligned}$$

Now use the definition of d_n in (1) and rearrange terms to obtain:

Theorem 2. (First-derivative form of Euler's summation formula). *For any function f with a continuous derivative on the interval* $[1, n]$ *we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \int_1^n (x - [x]) f'(x)\,dx + f(1). \tag{7}$$

The last two terms on the right represent the error made when the sum on the left is approximated by the integral $\int_1^n f(x)\,dx$. The formula is useful because f need not be positive or decreasing. In fact, f can be increasing or oscillating. Variants of this formula will be obtained as we attempt to deduce more precise information about the error.

The factor $x - [x]$ is a nonnegative function with period 1. If f' has a fixed sign (as it has when f is monotonic), the integral term in the error has the same sign as f'. To decrease the error it is preferable to multiply $f'(x)$ by a factor that changes sign so that some cancellation takes place in the integration. To introduce sign changes, we translate the function $x - [x]$ down by $\frac{1}{2}$ and consider the new function $x - [x] - \frac{1}{2}$ whose graph is shown in Figure 3. The integral term in the

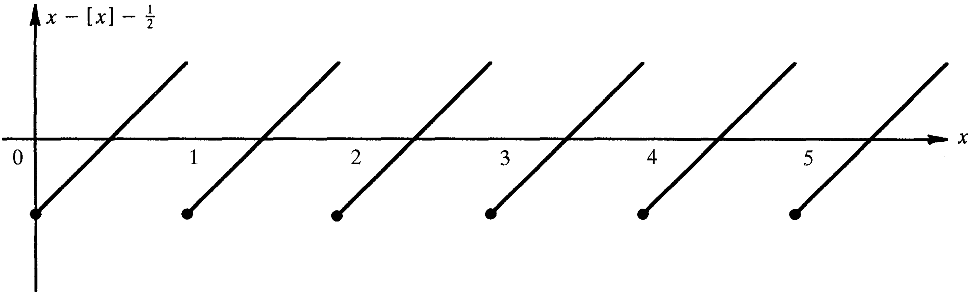

Figure 3. The periodic function $x - [x] - \frac{1}{2}$ changes sign.

error can now be written as

$$\int_1^n (x - [x]) f'(x)\, dx = \int_1^n \left(x - [x] - \frac{1}{2} \right) f'(x)\, dx + \frac{1}{2} \int_1^n f'(x)\, dx.$$

The last term is equal to $\frac{1}{2}\{f(n) - f(1)\}$. Using this in (7) we obtain the following variant of the first-derivative form of Euler's summation formula:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \int_1^n \left(x - [x] - \frac{1}{2} \right) f'(x)\, dx + \frac{1}{2}\{f(n) + f(1)\}. \quad (8)$$

Further variations will be obtained by repeated integration by parts in the second integral on the right of (8).

The factor $x - [x] - \frac{1}{2}$ has the value $-\frac{1}{2}$ when x is an integer. We modify this factor slightly to make it vanish at the integers, a property that is desirable when we integrate by parts. To do this we introduce $P_1(x)$, the *first Bernoulli function*:

$$P_1(x) = \begin{cases} x - [x] - \dfrac{1}{2} & \text{if } x \neq \text{integer} \\ 0 & \text{if } x = \text{integer}. \end{cases} \quad (9)$$

The error integral does not change if the factor $x - [x] - \frac{1}{2}$ is replaced by $P_1(x)$ because the two factors differ only at the integers. Therefore (8) can be written as

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \int_1^n P_1(x) f'(x)\, dx + \frac{1}{2}\{f(n) + f(1)\}. \quad (10)$$

Note the contrast between (10) and (3), which explicitly displays the generalized Euler's constant $C(f)$. To make (10) resemble (3) more closely, we assume that the improper integral $\int_1^\infty P_1(x) f'(x)\, dx$ converges. Then we can write

$$\int_1^n P_1(x) f'(x)\, dx = \int_1^\infty P_1(x) f'(x)\, dx - \int_n^\infty P_1(x) f'(x)\, dx,$$

and (10) takes the form

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + C(f) + E_f(n), \quad (11)$$

where

$$C(f) = \frac{1}{2}f(1) + \int_1^{\infty} P_1(x)f'(x)\,dx \tag{12}$$

and

$$E_f(n) = \frac{1}{2}f(n) - \int_n^{\infty} P_1(x)f'(x)\,dx.$$

Eq. (11) has exactly the same form as (3), but (11) is more general because f is not required to be positive or monotonic. The only restrictions on f are continuity of f' and convergence of the improper integral

$$\int_1^{\infty} P_1(x)f'(x)\,dx. \tag{13}$$

The improper integral in (13) converges if, and only if,

$$\lim_{n\to\infty} \int_n^{\infty} P_1(x)f'(x)\,dx = 0. \tag{14}$$

A sufficient condition for convergence is that $\int_1^{\infty}|f'(x)|\,dx$ converges, or equivalently, that

$$\lim_{n\to\infty} \int_n^{\infty} |f'(x)|\,dx = 0. \tag{15}$$

To see this, note that the Bernoulli function $P_1(x)$ is bounded; in fact, Figure 3 shows that $|P_1(x)| \le \frac{1}{2}$ for all x, so (14) follows from (15).

Example. When $f(x) = 1/x$ we have $f'(x) = -1/x^2$ and

$$\int_n^{\infty} |f'(x)|\,dx = \int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}.$$

Therefore (15) is satisfied and (12) expresses Euler's constant as an integral:

$$C = \frac{1}{2} - \int_1^{\infty} \frac{P_1(x)}{x^2}\,dx.$$

4. FURTHER ANALYSIS OF THE ERROR TERM. Alternate forms of both the error term and the formula for the generalized Euler's constant can be obtained by repeated integration by parts. First we introduce a new function $P_2(x)$ whose derivative is $2P_1(x)$ at all noninteger values of x. The factor 2 is used so that $P_2(x)$ is the second Bernoulli periodic function that appears in Euler's summation formula. Therefore we require that

$$P_2(x) = 2\int_0^{x} P_1(t)\,dt + c, \tag{16}$$

where c is a constant to be specified later. The function P_2 is quadratic on the interval $[0, 1]$. In fact, $P_2(x) = x^2 - x + c$ if $0 \le x \le 1$. Its graph is a parabolic arc joining the points $(0, c)$ and $(1, c)$. Outside this interval the graph (shown in Figure 4) consists of horizontal translations of this parabolic arc because P_2 has period 1. To see this, we use the fact that P_1 has period 1 and that $\int_0^1 P_1(t)\,dt = 0$, which implies that $\int_a^b P_1(t)\,dt = 0$ for any interval $[a, b]$ of length 1. Therefore

$$P_2(x+1) - P_2(x) = 2\int_x^{x+1} P_1(t)\,dt = 0.$$

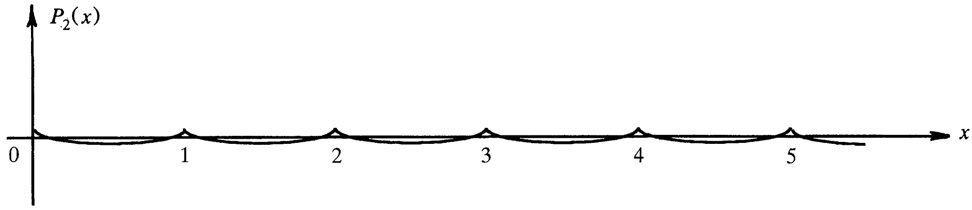

Figure 4. Graph of $P_2(x) = 2\int_0^x P_1(t)\,dt + c$.

Because of periodicity, P_2 has the constant value $c = P_2(0)$ at the integers. Integration by parts shows that the integral in (10) is

$$\int_1^n P_1(x)f'(x)\,dx = \frac{1}{2}P_2(0)\{f'(n) - f'(1)\} - \frac{1}{2}\int_1^n P_2(x)f''(x)\,dx,$$

provided that f'' is continuous. Repeated integration by parts leads to the general form of Euler's summation formula, which involves higher order derivatives of f and higher order periodic Bernoulli functions that represent polynomials on the unit interval [0, 1]. To see exactly how the Bernoulli functions evolve in the process we follow the method of the foregoing section and integrate the periodic function $3P_2(t)$ from 0 to x to obtain another periodic function $P_3(x)$ whose derivative is $3P_2(x)$. To guarantee that the integrated function $P_3(x)$ is periodic with period 1 we need $\int_0^1 P_2(t)\,dt = 0$. This property governs the choice of the constant c in (16). The integral of the quadratic polynomial $x^2 - x + c$ from 0 to 1 is equal to $c - \frac{1}{6}$, so we choose $c = \frac{1}{6}$ and take

$$P_2(x) = 2\int_0^x P_1(t)\,dt + \frac{1}{6}.$$

Euler's summation formula can now be restated as follows:

Theorem 3. (Second-derivative form of Euler's summation formula). *For any function f with a continuous second derivative on the interval* $[1, n]$ *we have*

$$\sum_{k=1}^n f(k) = \int_1^n f(x)\,dx - \frac{1}{2}\int_1^n P_2(x)f''(x)\,dx + \frac{1}{2}P_2(0)\{f'(n) - f'(1)\} + \frac{1}{2}\{f(n) + f(1)\}. \qquad (17)$$

Moreover, if the improper integral $\int_1^\infty |f''(x)|\,dx$ *converges then we also have*

$$\sum_{k=1}^n f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n),$$

where

$$C(f) = \frac{1}{2}f(1) - \frac{1}{2}P_2(0)f'(1) - \frac{1}{2}\int_1^\infty P_2(x)f''(x)\,dx, \qquad (18)$$

and

$$E_f(n) = \frac{1}{2}f(n) + \frac{1}{2}P_2(0)f'(n) + \frac{1}{2}\int_n^\infty P_2(x)f''(x)\,dx. \qquad (19)$$

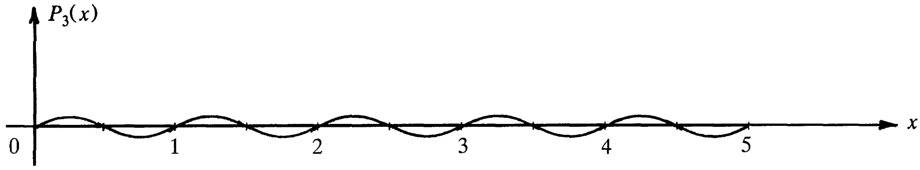

Figure 5. Graph of the periodic Bernoulli function $P_3(x) = 3\int_0^x P_2(t)\,dt$.

To improve the error estimate we integrate $P_2(t)$ from 0 to x and define the Bernoulli function $P_3(x) = 3\int_0^x P_2(t)\,dt$ so that $P_3'(x) = 3P_2(x)$. There is no need to add a constant in this case because, on the unit interval $[0, 1]$, $P_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, and $\int_0^1 P_3(t)\,dt = 0$. The function P_3 has period 1 because P_2 has period 1 and $\int_0^1 P_2(t)\,dt = 0$. The graph of P_3 is a bounded piecewise cubic curve, shown in Figure 5. Note that $P_3(x)$ vanishes at the integers. Integration by parts over $[1, n]$ gives us

$$\int_1^n P_2(x) f''(x)\,dx = -\frac{1}{3}\int_1^n P_3(x) f^{(3)}(x)\,dx,$$

provided $f^{(3)}$ is continuous. This equation, together with Theorem 3, gives a third-derivative form of Euler's summation formula in which the second integral on the right of (17) is replaced by $\frac{1}{3!}\int_1^n P_3(x) f^{(3)}(x)\,dx$. The corresponding changes in (18) and (19) are replacement of the integrals by $\frac{1}{3!}\int_1^\infty P_3(x) f^{(3)}(x)\,dx$ and $\frac{-1}{3!}\int_n^\infty P_3(x) f^{(3)}(x)\,dx$, respectively.

5. BERNOULLI NUMBERS AND THE GENERAL FORM OF EULER'S SUMMATION FORMULA. The strategy for obtaining a general version of Euler's summation formula is now evident. Starting with the Bernoulli periodic function $P_1(x)$ in (9) we introduce, in succession, periodic functions $P_2(x), P_3(x), \ldots$, with period 1, and a sequence of constants B_k such that

$$P_k(x) = k\int_0^x P_{k-1}(t)\,dt + B_k \quad \text{for } k \geq 2, \tag{20}$$

where each B_k is chosen so that

$$\int_0^1 P_k(t)\,dt = 0. \tag{21}$$

Periodicity implies that $P_k(0) = P_k(1)$, and (21) shows that each of these values is B_k. As already noted, on the closed interval $[0, 1]$ each function $P_k(x)$ is a polynomial of degree k when $k = 2$ or 3. [The case $k = 1$ is special; $P_1(x)$ is a linear polynomial $x - \frac{1}{2}$ only on the open interval $(0, 1)$ and is discontinuous at the endpoints.] It is clear (and easily proved by induction) that on the closed interval $[0, 1]$ the function defined by (20) is a polynomial of degree k if $k \geq 2$. We denote this polynomial by $B_k(x)$, the usual notation for *Bernoulli polynomials*. The first few are

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x,$$

$$B_6(x) = x^6 - \frac{3}{5}x^5 + \frac{1}{2}x^4 - \frac{1}{10}x^2 + \frac{1}{42}.$$

The *Bernoulli periodic functions* are periodic extensions of these polynomials given by $P_k(x) = B_k(x - [x])$. The constants $B_k = P_k(0) = P_k(1)$ are called *Bernoulli numbers*. The first few are

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30},$$

$$B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}.$$

Next we show that our definitions of Bernoulli numbers and polynomials are consistent with the usual definitions, provided we take $B_0(x) = 1$ and $B_0 = 1$. Our definition in (20) shows that the successive derivatives of these polynomials are

$$B_k'(x) = kB_{k-1}(x), B_k''(x) = k(k-1)B_{k-2}(x), \dots, B_k^{(r)}(x) = r!\binom{k}{r}B_{k-r}(x),$$

and hence

$$B_k^{(r)}(0) = r!\binom{k}{r}B_{k-r}(0) = r!\binom{k}{r}B_{k-r}. \tag{22}$$

On the other hand, the Taylor expansion of any polynomial $B_k(x)$ of degree k is given by $B_k(x) = \sum_{r=0}^{k} B_k^{(r)}(0)x^r/r!$, so (22) implies

$$B_k(x) = \sum_{r=0}^{k}\binom{k}{r}B_{k-r}x^r. \tag{23}$$

Taking $x = 1$ in (23) and noting that $B_k(1) = P_k(1) = B_k$ for $k \geq 2$, we find that (23) becomes

$$B_k = \sum_{r=0}^{k}\binom{k}{r}B_{k-r} \quad \text{for } k \geq 2.$$

This is the usual recursion formula for defining Bernoulli numbers (starting with $B_0 = 1$), and (23) is one of the standard ways of defining Bernoulli polynomials in terms of Bernoulli numbers. Consequently, the numbers and polynomials that appear in our treatment are the usual Bernoulli numbers and Bernoulli polynomials that appear in the literature; see [**1**, p. 265], [**2**, p. 251], or [**5**, pp. 160–163].

It is well known that the Bernoulli numbers B_k with odd index $k \geq 3$ are zero, so only Bernoulli numbers with even index appear in the general form of Euler's summation formula. It is also known [**8**, p. 533] that on the interval $[0, 1]$ the Bernoulli polynomials satisfy the following inequalities for $k \geq 1$:

$$|B_{2k}(x)| \leq |B_{2k}| \quad \text{and} \quad |B_{2k+1}(x)| \leq (2k+1)|B_{2k}|.$$

The method we have outlined leads to the following odd-order derivative version of Euler's summation formula. A proof is easily given by induction on the order $2m + 1$.

Theorem 4. (General form of Euler's summation formula). *For any function f with a continuous derivative of order $2m + 1$ on the interval $[1, n]$ we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(x)f^{(2m+1)}(x)\,dx$$
$$+ \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\{f^{(2r-1)}(n) - f^{(2r-1)}(1)\} + \frac{1}{2}\{f(1) + f(n)\}. \tag{24}$$

Moreover, if the improper integral $\int_1^\infty |f^{(2m+1)}(x)|\,dx$ *converges then we also have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n), \tag{25}$$

where

$$C(f) = \frac{1}{2}f(1) - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(1) + \frac{1}{(2m+1)!}\int_1^\infty P_{2m+1}(x) f^{(2m+1)}(x)\,dx, \tag{26}$$

and

$$E_f(n) = \frac{1}{2}f(n) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) - \frac{1}{(2m+1)!}\int_n^\infty P_{2m+1}(x) f^{(2m+1)}(x)\,dx. \tag{27}$$

Example. When $f(x) = 1/x$ we have $f^{(2m+1)}(x) = -(2m+1)!/x^{2m+2}$, and (26) gives the following expression for the classical Euler's constant:

$$C = \frac{1}{2} + \frac{B_2}{2} + \frac{B_4}{4} + \cdots + \frac{B_{2m}}{2m} - \int_1^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}\,dx. \tag{28}$$

The corresponding error term (27) becomes

$$E_f(n) = \frac{1}{2n} - \frac{B_2}{2n^2} - \frac{B_4}{4n^4} - \cdots - \frac{B_{2m}}{2mn^{2m}} + \int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}\,dx. \tag{29}$$

One is tempted to let $m \to \infty$ in (28) and obtain an infinite series for Euler's constant. However, the integral in (28) does not tend to 0 as $m \to \infty$ and, in fact, it can be shown that the infinite series $\Sigma B_{2k}/(2k)$ diverges rapidly [see **6**, p. 529], so (28) is not very useful for calculating C. Nevertheless, as we show in the next section, (25) and (27) can be used to calculate C very accurately.

6. CALCULATION OF EULER'S CONSTANT. We use Euler's summation formula to calculate the first 7 digits in Euler's constant. Take $f(x) = 1/x$ in (25) and rewrite it as

$$C = \sum_{k=1}^{n} \frac{1}{k} - \log n - E_f(n), \tag{30}$$

where $E_f(n)$ is given by (29). Taking $m = 3$ in (29) we find

$$E_f(n) = \frac{1}{2n} - \frac{B_2}{2n^2} - \frac{B_4}{4n^4} - \frac{B_6}{6n^6} + \int_n^\infty \frac{P_7(x)}{x^8}\,dx$$
$$= \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \int_n^\infty \frac{P_7(x)}{x^8}\,dx.$$

Using the inequality $|P_7(x)| \le 7|B_6| = \frac{1}{6}$, we get

$$\left|\int_n^\infty \frac{P_7(x)}{x^8}\,dx\right| \le \frac{1}{6}\int_n^\infty \frac{1}{x^8}\,dx = \frac{1}{42n^7},$$

and (30) can be written in the form

$$C = \sum_{k=1}^{n} \frac{1}{k} - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} + E(n), \qquad (31)$$

where $0 < |E(n)| \leq 1/42n^7$. Using a hand calculator that displays 12 digits we find $\sum_{k=1}^{10} k^{-1} = 2.92896825381$ and $\log 10 = 2.30258509299$. If $n = 10$ the sum of the error term $E(n)$ plus the term with $252n^6$ in the denominator in (31) is too small to influence the seventh digit. Neglecting these terms and retaining 8 digits in the calculation we find

$$\begin{aligned} C &\doteq 2.92896825 - 2.30258509 - \frac{1}{20} + \frac{1}{1200} - \frac{1}{1200000} \\ &= 0.62638316 - 0.05000000 + 0.00083333 - 0.00000083 \\ &= 0.57721566 \end{aligned}$$

This calculation, using $m = 3$ and $n = 10$ in (29) and (30), which guarantees 7 decimal places, actually gives the first 8 correct digits of C. Knuth [**7**] used (29) and (30) with $m = 250$ and $n = 10{,}000$ to calculate the value of C to 1,271 decimal places.

This note outlines only one application of Euler's summation formula. Others can be found in Knopp's treatise [**6**]. One of them uses the increasing function $f(x) = \log x$ to derive Stirling's asymptotic formula for the logarithm of $n!$. Euler's summation formula and its relation to Bernoulli numbers and polynomials provides a treasure trove of interesting enrichment material suitable for elementary calculus courses.

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