The Isoperimetric Problem on Surfaces

Hugh Howards; Michael Hutchings; Frank Morgan

Stable URL:

The American Mathematical Monthly is currently published by Mathematical Association of America.

The Isoperimetric Problem on Surfaces

Hugh Howards, Michael Hutchings, and Frank Morgan

1. INTRODUCTION. The isoperimetric problem on a surface is to enclose a given area with the shortest possible curve. The classical isoperimetric theorem asserts that in the plane the unique solution is a circle. On curved surfaces the isoperimetric problem is harder and much remains open. Even on the simplest paraboloid the "obvious" solution was proved only in 1996 by Benjamini and Cao ([**2**, Thms. 5, 8]; see also [**24**, Prop. 7], [**22**, Thm. 3.1], [**30**, Thm. 1], [**29**], [**26**]):

Theorem 1.1 (Benjamini and Cao). *The unique least-perimeter way to enclose given area in the paraboloid of revolution*

$$P = \left\{ z = x^2 + y^2 \right\} \subset \mathbf{R}^3 \tag{1.1}$$

is a horizontal circle $\{z = c\}$.

   This article gives our three favorite proofs of the classical isoperimetric theorem in the plane and then presents some recent results on other surfaces, including a new proof for the paraboloid. Section 2 uses an amazingly simple symmetry argument to show that a nice minimizer must be a circle. Unfortunately this approach needs to assume that a nice minimizer exists. Section 3 gives a very simple, complete proof without assuming a nice minimizer exists, following the undergraduate thesis of Howards [**15**]. Section 4 provides another complete proof, a slight twist on a magical proof of Gromov [**10**].

   In general surfaces the existence of a nice, one-component perimeter-minimizing curve has been astonishingly problematic. Fortunately a relatively easy approach is now available from [**12**], as explained in Section 5. One has to allow the curve to bump up against itself.

   Sections 6–8 solve the isoperimetric problem for cylinders, cones, flat tori, and Klein bottles. Section 9 treats the paraboloid and certain other surfaces of revolution. Section 10 discusses hyperbolic surfaces.

   This work was partly inspired by a more difficult question we heard from J. C. C. Nitsche about the soap film between a large wire boundary and a small, moveable loop of thread. The thread wants to position itself to minimize the area of the soap film outside it. If the thread were constrained to lie in a fixed surface bounded by the wire (which unfortunately is not the case), then the thread would want to be an isoperimetric curve in that surface.

   Osserman [**23**] provides a marvelous survey on the isoperimetric inequality.

2. THE CIRCLE IN THE PLANE, ASSUMING SMOOTH EXISTENCE. We assume that there is a compact minimizer C among smooth curves of finitely many components and enclosed area π, and use symmetry to prove it must be a single round unit circle; existence is a nontrivial assumption, a fact overlooked by some early workers. The proof uses a symmetry argument we heard from Brian White and Luen-fai Tam, who thought it originated with Blaschke (see [**9**, Thm. 3.4], [**17**, Thm. 5.3], and [**16**, §2]); we have been unable to trace its origin and would be grateful to anyone who could help.

Suppose C is not a round circle. Take a horizontal line splitting the enclosed area in half. Each half must have the same length, or the shorter half, together with its reflection, would be shorter than C. Replacing C by half plus its reflection if necessary, we may assume that C is symmetric across the horizontal line. Similarly we may assume that C is symmetric across a vertical line. We may assume the lines meet at the origin. Now C is symmetric under the composition of the two reflections, i.e., under 180-degree rotation around the origin. Hence every line through the origin splits the area in half. C must meet every line through the origin orthogonally; otherwise, one half of C, together with its reflection, would not be convex, and its convex hull would have less perimeter and more area. It follows that C consists of circles about the origin. A single circle is best. We conclude that the original C is a round circle.

This argument can be generalized to prove that a round hypersphere is perimeter-minimizing for given volume in $\mathbf{R}^n$, in the round sphere $\mathbf{S}^n$, and in hyperbolic space $\mathbf{H}^n$. More generally, it shows that a minimizing cluster of k bubbles enclosing $k < n$ prescribed volumes in $\mathbf{R}^n$ has $\mathbf{O}(n - k + 1)$ symmetry, assuming known but difficult existence and regularity [16, Thm. 2.6]. It played an essential role in the recent proof by Hass, Hutchings, and Schlafly of the equal volumes case of the still open Double Bubble Conjecture, which says that the familiar standard double soap bubble is the least-area way to enclose and separate two given volumes of air ([11], [16], [14], [18], [13]).

3. THE CIRCLE IN THE PLANE, WITHOUT ASSUMING EXISTENCE. To prove that the circle is perimeter-minimizing (but not necessarily uniqueness), by approximation it suffices to show that the shortest n-gon enclosing given area is the regular n-gon. In his undergraduate thesis, Howards [15] gave the following geometric proof free of variational calculus, including ideas that we have since traced back to Zenodorus about 200 BC, Steiner in 1838 ([27], [28, p. 105 and Fig. 6]), and Courant and Hilbert [6, p. 166]; see the interesting "A history of the classical isoperimetric problem" by Porter [25] and Bonnesen and Fenchel [5, §57].

By compactness, there is a shortest n-gon in the $2n$-dimensional space of vertices. It is convex. Consider two adjacent sides, which determine a triangle, and the line L through the common vertex and parallel to the third side of the triangle. These two sides must constitute the shortest path to L and back, since all such constructions yield triangles of the same area. The first side, together with the reflection of the second across L, must form a straight line. Hence the two sides have the same length. Therefore the n-gon is equilateral.

To prove that the equilateral n-gon is regular, we begin with n even. For opposite vertices P, Q, the line PQ must have the same area above as below, or a reflection of the larger half would enclose more area (or, scaled down, the same area with less length). For an intermediate vertex M, the angle PMQ must be 90°, or replacing it with a 90° angle and reflecting as in Figure 3.1 would increase the area enclosed. Therefore the n-gon is inscribed in a circle and must be regular.

Finally suppose n is odd. A regular $2n$-gon comes from putting little triangles on the sides of the regular n-gon. If a perimeter-minimizing n-gon, known to be equilateral, had more area than a regular n-gon with the same sides, putting those little triangles on its sides would yield a non-regular $2n$-gon with more area than the regular $2n$-gon, the final contradiction.

This completes the proof that the circle is perimeter minimizing. In fact, now that we know that a minimizer exists, we can use the above arguments to prove
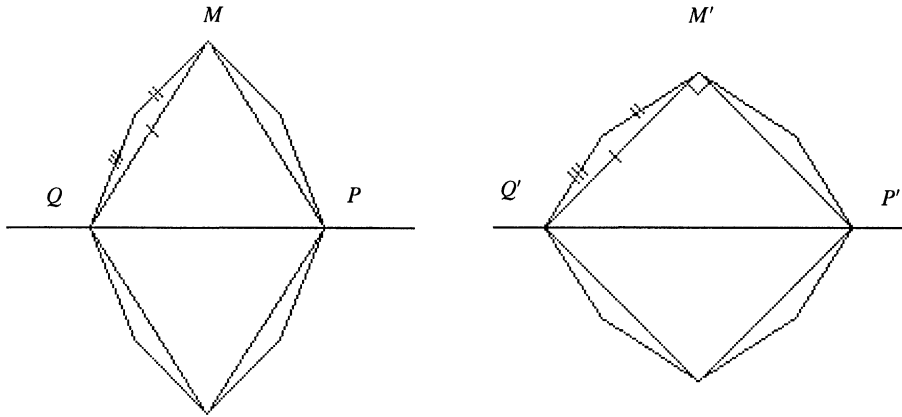
Figure 3.1. The angle PMQ must be 90°, or replacing it with a 90° angle and reflecting would increase the area enclosed.

uniqueness. Consider any minimizer. It must be convex. As above, a line bisecting its perimeter must bisect the area, and any inscribed angle must be 90°. Therefore, the minimizer must be a circle.

4. THE CIRCLE IN THE PLANE, ANOTHER PROOF WITHOUT ASSUMING EXISTENCE. Gromov ([10]; see [3, 12.11.4] or [20, 10.5]) gave a proof of the isoperimetric theorem in $\mathbf{R}^n$ by direct comparison. The strategy in $\mathbf{R}^2$, for example, is to find a vectorfield v on any competing region R of area π with smooth boundary C and outward unit normal $\mathbf{n}$ such that

$$\text{div}(v) \geq 2, \tag{4.1}$$

$$v \cdot \mathbf{n} \leq 1, \tag{4.2}$$

with equality everywhere only if R is a disc. If such a v can be found, then the isoperimetric inequality follows immediately from Stokes' theorem:

$$\text{length}(C) \geq \int_C v \cdot \mathbf{n} = \int_R \text{div}(v) \geq 2\,\text{area}(R) = 2\pi,$$

with equality only if R is a disc.

The Gromov proof finds such a v by a very clever construction, but the resulting v is not canonical. We now show that there is a canonical such v when $n = 2$.

The canonical v is the negative of the *gravitational field* induced by a substance of constant density filling the region R. More precisely,

$$v(x) = \frac{1}{\pi} \int_{y \in R} \frac{x - y}{|x - y|^2} \, dy.$$

By the two-dimensional analog of Gauss's law, $\text{div}(v) = 2$ in R, so it now suffices to prove (4.2).

Fix a point x in the boundary, and choose polar coordinates (r, θ) around x so that $\mathbf{n}$ points in the direction $\theta = \pi$. Then

$$v(x) \cdot \mathbf{n} = \frac{1}{\pi} \int_{y \in R} \frac{\cos \theta}{r} \, dy.$$

Since the area of R is fixed, this integral is maximized if we put the points of R where $(\cos \theta)/r$ is largest. The level sets of $(\cos \theta)/r$ are circles tangent to C at x, with smaller circles giving larger values of $(\cos \theta)/r$. So clearly a disc of the given area uniquely maximizes the integral, completing the proof.

5. PROOF OF EXISTENCE OF NICE LEAST-PERIMETER ENCLOSURES.

In the Euclidean plane and in other special cases where all candidates can be convexified, the existence of a (convex) region of least perimeter and prescribed area follows from Blaschke's selection theorem ([4, p. 38], [8, Chapt. 4]). A general smooth Riemannian surface S requires a more general argument. There must be some hypothesis to prevent the solution from disappearing to infinity as in Figure 5.1. Suppose for now that S is a compact surface, perhaps with convex boundary.
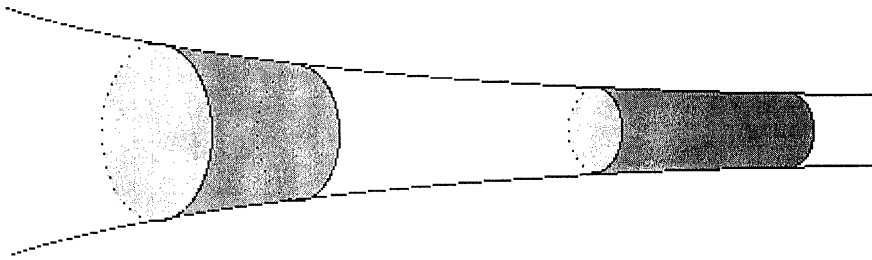
Figure 5.1. In the surface of revolution generated by $y = 1/x$, for any given area, there is a sequence of annuli disappearing to infinity with perimeters going to 0.

For the moment we restrict to images of the unit circle parametrized by arclength. Later we consider curves of several components. Then compactness properties of Lipschitz functions (Ascoli-Arzela Theorem) immediately produce a minimizer. The only problem is that in theory the limit might bump up against itself too wildly to permit the standard variational argument that it has constant geodesic curvature. The solution may actually bump up against itself, as in the cylinder of Figure 5.2. This technical difficulty delayed for 75 years the completion of Poincare's proof that every smooth convex sphere contains a simple closed geodesic. In 1982 C. Croke [7] gave a complete proof by minimizing a combination of length and energy in a class of piecewise geodesic curves.

Figure 5.2. Some least perimeter enclosures on the cylinder bump up against themselves.

More recently, Hass and Morgan ([12]; see also [22, Lemma 2.2]) have provided a very simple approach to more general existence and regularity using local convexification. Away from the boundary of S, a minimizing enclosure is an embedded curve of constant geodesic curvature κ_0, except possibly for finitely many geodesic arcs or isolated points where it bumps up against itself but remains C^1. Even at the boundary of S the curve remains C^1 and the geodesic curvature

satisfies $\kappa \le \kappa_0$ (weakly). If for bounded area the curve is allowed a large number of components enclosing disjoint regions, no curve bumps itself on the inside and $\kappa \le \kappa_0$ everywhere. If the curve is allowed several nested components enclosing multiply connected regions, it never bumps itself. A word on the proof: if local convexification causes two pieces of curve to cross, the longer one is rerouted along the shorter. This process reduces length unless the curves were convex to begin with. Given convexity, standard variational arguments prove the rest.

In many noncompact manifolds, such as the Euclidean or hyperbolic plane, one can work inside a large convex set. Hyperbolic surfaces and other surfaces can have thin cusps to infinity with nonconvex truncations, but as long as the area of the cusp is finite, any sequence of curves going off to infinity has area going to 0 and may be discarded.

Existence and some regularity hold as well for clusters in $\mathbf{R}^2$ (enclosing and separating several regions of prescribed areas [21]) and in general dimensions by the techniques of geometric measure theory [19, Chapt. 13]. In higher dimensions you cannot hope to prescribe the topological type; for example, regions connected by thin tubes can disconnect in the limit. Even for curves in the plane, such general techniques do not have the topological control we need.

6. CIRCULAR CYLINDERS. *On the cylinder $\{x^2 + y^2 = a^2\} \subset \mathbf{R}^3$, the least-perimeter enclosure of area A is a small (round) circle for $A \le 4\pi a^2$ and two horizontal circles for $A \ge 4\pi a^2$.*

Proof: We know that any solution consists of closed curves of constant curvature. If one curve is homotopically trivial and hence is a small round circle, it is the only one, or it could be translated to touch another and contradict regularity. If all the curves are homotopically nontrivial, there must be at least two of them to enclose area, and two horizontal circles are best. The transition occurs when the circumference of the small circle $2\sqrt{\pi A}$ equals the length of two horizontal circles $4\pi a$, i.e., $A = 4\pi a^2$.

7. FLAT TORI AND KLEIN BOTTLES (Howards [15, Thm. 3.1]). *Let S be a flat torus or Klein bottle with shortest closed geodesic of length a. Given $0 < A < \text{area } S$, the least-perimeter region of area A is*

(1) *a circular disc if $0 < A \le a^2/\pi$;*
(2) *a band (possibly Möbius) with geodesic boundary if $a^2/\pi \le A \le \text{area } S - a^2/\pi$;*
(3) *the complement of a circular disc if area $S - a^2/\pi \le A \le \text{area } S$.*

Proof: Any solution consists of closed curves of constant curvature. As in the argument in Section 6, if one is homotopically trivial and therefore a small round circle, it is the only one or it could be translated to touch another and contradict regularity. If all the components are nontrivial, for any given area the perimeter is uniquely minimized by a single geodesic band with perimeter $2a$. The transitions between types occur when the circle has circumference $2a$.

Remark. The round sphere and round projective plane may be treated by similar arguments [15, Thms. 4.1, 5.1] or by the methods of Section 9 [22, Thms. 3.1, 3.3].

8. CIRCULAR CONES. *On the circular cone $\{z = a\sqrt{x^2 + y^2}\} \subset \mathbf{R}^3$, the least-perimeter enclosure of area A is a horizontal circle.*
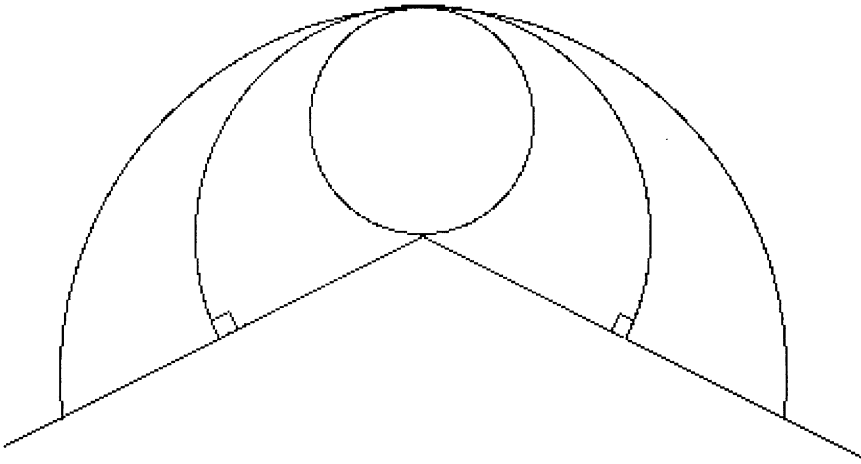
Figure 8.1. A constant-curvature curve about the vertex of the cone must be a circle of smaller circumference than a planar circle of the same area.

Proof: If any component does not encircle the vertex, it must be the only component (or it could be translated to touch another component, contradicting regularity), and hence it must be a circle of length $2\sqrt{\pi A}$. Consider a constant-curvature curve that encircles the vertex. It must be symmetric about the line through the vertex and a point most distant from the vertex, as in Figure 8.1, so it must be a horizontal circle. Clearly a single horizontal circle would have less perimeter than several. Since one horizontal circle has length less than $2\sqrt{\pi A}$, it must be the minimizer.

Remark. Actually for a simply-connected domain D on a surface with Gauss curvature K, the perimeter L satisfies

$$L^2 \geq 4\pi A - 2A \int_D \max\{K, 0\},$$

with equality for the singular limit case of the cone [**23**, Thm. 4.3].

9. THE PARABOLOID AND OTHER SURFACES OF REVOLUTION. Sections 9 and 10 provide some new examples. The following theorem and corollary include the paraboloid. The proof integrates the Gauss-Bonnet theorem.

Theorem 9.1 ([**24**, Prop. 7], [**22**, Thm. 2.1], [**29**]). *Consider the plane with smooth, rotationally symmetric, complete metric such that the Gauss curvature is a strictly decreasing function of the distance from the origin. Then the unique length-minimizing simple closed curve enclosing a given area is a circle centered at the origin.*

By Section 5, inside a surface of finite area or inside a large convex ball B, for bounded area, there is a minimizer among C^1 curves of $m \leq m_0$ components, enclosing m disjoint discs. Away from ∂B, it is an embedded curve of constant geodesic curvature κ_0, except possibly for finitely many geodesic arcs or isolated points where it bumps up against itself. If m_0 is large, the curvature $\kappa \leq \kappa_0$ everywhere.

A short proof of the following standard technical lemma is in [**20**, §9.7, p. 112] (cf. [**22**, Lemma 2.3]). The idea is that the rate of change of the perimeter is essentially the geodesic curvature, which is controlled by the Gauss-Bonnet theorem.

Lemma 9.2. *Let $L(A)$ denote the least perimeter of $m \leq m_0$ discs of total area A. Then $L(A)$ is differentiable almost everywhere and*

$$L(A)^2 \geq 2 \int_0^A LL'.$$

Proof of Theorem 9.1 [**22**]: In the surface or inside a large convex ball, for m_0 large, let $L(A)$ denote the length of a shortest curve of $m \leq m_0$ components enclosing m disjoint discs of total area A. First we claim that if L is differentiable at A, then $L'(A)$ is the geodesic curvature κ_0. One geometric interpretation of geodesic curvature is the rate of change of length with respect to area under perturbations of the given minimizer [**20**, Chapt. 2]. Hence for $\Delta A > 0$, the new minimizers must do at least as well as perturbations of the old one, and $L'(A) \leq \kappa_0$. Similarly for $\Delta A < 0$, $-L'(A) \leq -\kappa_0$. Therefore $L'(A) = \kappa_0$.

Now Gauss-Bonnet tells us that the total Gauss curvature of the enclosed region equals

$$2\pi m - \int \kappa \geq 2\pi - L(A)\kappa_0 = 2\pi - L(A)L'(A).$$

Let $G(A)$ denote the total Gauss curvature of a disc of area A centered at the origin. Since the Gauss curvature is a decreasing function of radius, any other region with the same area must have less total Gauss curvature. So we have

$$2\pi - L(A)L'(A) \leq G(A), \tag{9.1}$$

and

$$L(A)L'(A) \geq 2\pi - G(A).$$

By Lemma 9.2, integration from $A = 0$ to A_1 yields

$$L(A)^2 \geq 2 \int L(A)L'(A) \geq 4\pi A_1 - 2 \int G(A). \tag{9.2}$$

This inequality is sharp for a circle centered at the origin, as we can see by integrating the Gauss-Bonnet formula for circles centered at the origin with area A from 0 to A_1. Hence equality holds in (9.2), L is differentiable everywhere, and equality holds in (9.1). Therefore a minimizer encloses Gauss curvature $G(A)$ and must be a circle about the origin.

The following general extension to several, perhaps multiply connected regions is deduced in [**22**, Thm. 3.1]. Here we give a proof for the easy case of positive Gauss curvature, which includes the paraboloid.

Corollary 9.3. *Among unions of disjoint, perhaps multiply connected regions, a perimeter minimizer exists and is a (round) disc, disc complement, or annulus about the origin. If the Gauss curvature is positive or the total Gauss curvature of every compact region is less than 2π, then the minimizer is a disc.*

Proof for the case of positive Gauss curvature: By the Gauss-Bonnet theorem, the perimeter $P(r)$ and geodesic curvature $\kappa(r)$ of a circle about the origin of radius r

bounding a disc D satisfy

$$P' = \kappa P = 2\pi - \iint_D K. \tag{9.3}$$

The total Gauss curvature is at most 2π, since otherwise eventually $P' \le -a < 0$ and P hits 0. By (9.3), κ is positive and decreasing. Consider any collection of simple closed curves enclosing area A_1. By discarding any curves inside others, enclose area $A_2 \ge A_1$. By Theorem 9.1, each curve alone would be shortest if a circle about the origin. Since $\kappa = dP/dA$ is decreasing, one single circle about the origin is best. Since $A_1 \le A_2$, the circle of area A_1 is best of all.

10. HYPERBOLIC MANIFOLDS. We consider geometrically finite complete hyperbolic surfaces (curvature -1). Such surfaces may be compact or have finitely many ends: cusps (with exponentially shrinking thickness and finite area) or flared ends (asymptotic to the hyperbolic plane).

Theorem 10.1 [1, Thm. 2.2]. *Let S be a hyperbolic surface. For given area $0 < A < \text{area}(S)$, a perimeter-minimizing system of embedded rectifiable curves bounding a region of area A consists of curves of the same constant curvature of one of four types*:

   (I) *a circle,*
  (II) *horocycles around cusps,*
 (III) *two "neighboring curves" at constant distance from a geodesic, bounding an annulus or complement,*
 (IV) *geodesics or single neighboring curves.*

The total perimeter L satisfies

$$L \le \sqrt{A^2 + 4\pi A}, \tag{10.1}$$

with equality for a circle of area A. If S has at least one cusp, then cases (I) and (III) do not occur and $L \le A$; if moreover $A < \pi$, then a minimizer consists of horocycles bounding neighborhoods of an arbitrary collection of cusps and has perimeter $L = A$.

Proof sketch: The constant-curvature curves on a hyperbolic surface are circles bounding discs ($\kappa > 1$) or the complement ($\kappa < -1$), horocycles around cusps ($\kappa = 1$) or the complement ($\kappa = -1$), and constant-curvature curves around necks ($|\kappa| < 1$, including the geodesics around the middle of necks with $\kappa = 0$).

A minimizer cannot have more than one circle, since sliding one until it hits another (or itself) would contradict regularity. Since for other types, $dL/dA = \kappa$ is less than it is for a circle, (10.1) always holds, and there is an $A_0 \ge 0$ such that if $A < A_0$ the minimizer is a circle, while if $A > A_0$ it is not a circle and (for $\Delta A > 0$)

$$\Delta L / \Delta A < 1. \tag{10.2}$$

Now a computation shows that an annulus (or complement) as in Case (III) must occur alone, or an operation such as discarding it would contradict (10.2). Therefore the minimizer falls into one of the four asserted cases.

Henceforth assume S has a cusp. Case (I) cannot occur, because sliding the circle out the cusp until it hits itself would contradict regularity. Hence the minimizer always has $|\kappa| \le 1$, and always $L(A) \le A$. A computation shows that Case (III) cannot occur.

Finally assume $A < \pi$. We claim *there is no minimizer with* $-1 \le \kappa < 1$ *and length* $L \le A$, so $-A + \kappa L < 0$. Otherwise, applying Gauss-Bonnet to the enclosed region yields

$$2\pi\chi = -A + \kappa L < 0,$$

$\chi \le -1$, $-A + \kappa L \le -2\pi$, $\kappa L \le -\pi$, $\kappa < 0$, $L \ge \pi > A$, a contradiction.

The remaining possibilities, systems of curves with $\kappa = 1$, consist of horocycles bounding cusp neighborhoods. Since $\kappa = 1$, as you slide a horocycle out a cusp $dL/dA = 1$, and therefore its length equals the area of the cusp neighborhood. By the claim, such systems remain minimizing as long as they exist, either for all $A < \pi$ or until they bump up against themselves at some $A_1 < \pi$. If one bumps, by regularity the minimizer has perimeter less than A_1, contradicting the claim and proving the theorem.

REFERENCES

1. Colin Adams and Frank Morgan, Isoperimetric curves on hyperbolic surfaces, *Proc. Amer. Math. Soc.*, to appear.
2. Itai Benjamini and Jianguo Cao, A new isoperimetric theorem for surfaces of variable curvature, *Duke Math. J.* 85 (1996) 359–396.
3. M. Berger, *Geometry II*, Springer-Verlag, New York, 1977.
4. W. Blaschke, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Berlin, 1923.
5. T. Bonnesen and W. Fenchel, *Theory of Convex Bodies* (English translation of *Theorie der Konvexen Körper*), BCS Associates, Moscow, Idaho, 1987.
6. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I, Wiley & Sons, NY, 1953.
7. C. Croke, Poincare's problem on the shortest closed geodesic on a convex hypersurface, *J. Differential Geom.* 17 (1982) 595–634.
8. Harold Gordon Eggleston, *Convexity*, Cambridge University Press, Cambridge, 1966.
9. Joel Foisy, Soap bubble clusters in $\mathbf{R}^2$ and $\mathbf{R}^3$, undergraduate thesis, Williams College, 1991.
10. M. Gromov, Isoperimetric inequalities in Riemannian manifolds, Appendix I to Vitali D. Milman and Gideon Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics, No. 1200, Springer-Verlag, New York, 1986.
11. Joel Hass, Michael Hutchings, and Roger Schlafly, The double bubble conjecture, *Electron. Res. Announc. Amer. Math. Soc.* 1 (1995) 98–102.
12. Joel Hass and Frank Morgan, Geodesics and soap bubbles in surfaces, *Math. Z.* 223 (1996) 185–196.
13. Joel Hass and Roger Schlafly, Bubbles and double bubbles, *American Scientist*, Sept.–Oct., 1996, pp. 462–467.
14. Joel Hass and Roger Schlafly, Double bubbles minimize, *Ann. of Math.*, to appear.
15. Hugh Howards, Soap bubbles on surfaces, undergraduate thesis, Williams College, 1992.
16. Michael Hutchings, The structure of area-minimizing double bubbles, *J. Geom. Anal.* 7 (1997) 285–304.
17. Frank Morgan, Clusters minimizing area plus length of singular curves, *Math. Ann.* 299 (1994) 697–714.
18. Frank Morgan, The Double Bubble Conjecture, *FOCUS*, Math. Assn. Amer., December, 1995.
19. Frank Morgan, *Geometric Measure Theory: a Beginner's Guide*, Academic Press, second edition, Boston, 1995.
20. Frank Morgan, *Riemannian Geometry: a Beginner's Guide*, A. K. Peters, Ltd., second edition, Natick, 1998.
21. Frank Morgan, Soap bubbles in $\mathbf{R}^2$ and in surfaces, *Pacific J. Math.* 165 (1994) 347–361.
22. Frank Morgan, Michael Hutchings, and Hugh Howards, The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature, *Trans. Amer. Math. Soc.*, to appear.
23. Robert Osserman, The isoperimetric inequality, *Bull. Amer. Math. Soc.* 84 (1978) 1182–1238.
24. Pierre Pansu, Sur la régularité du profil isopérimétrique des surfaces riemanniennes compactes, *Ann. Inst. Fourier* 48 (1998) 247–264.

25. Thomas Isaac Porter, A history of the classical isoperimetric problem, in *Contributions to the Calculus of Variations 1931–1932*, Univ. of Chicago Press, Chicago, 1933, pp. 475–520.
26. Manuel Ritoré, Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces, preprint (1998).
27. J. Steiner, Einfache Beweise der isoperimetrischen Hauptsatze, *Crelle's J.* 18 (1838) 286–287.
28. J. Steiner, Sur le maximum et minimum des figures dans le plan, sur le sphere, et l'espace en general, *Crelle's J.* 24 (1842) 93–152, 189–250 (figures separate at end of fascicle).
29. Peter Topping, The isoperimetric inequality on a surface, preprint (1997).
30. Peter Topping, Mean curvature flow and geometric inequalities, *J. Reine Angew. Math.* 503 (1998) 47–61.

The three authors all spent time at Williams College, where Howards and Hutchings participated in Morgan's NSF undergraduate research Geometry Group and Howards wrote his undergraduate thesis.

Howards, who went to Williams and UC San Diego, is assistant professor of Mathematics at Wake Forest University. Hutchings, who went to Harvard, is Szego Assistant Professor of Mathematics at Stanford. Morgan, who went to MIT and Princeton, is Meenan Third Century Professor of Mathematics at Williams College. In January, 1993, he received one of the first MAA national awards for distinguished teaching.

Morgan has a biweekly Math Chat column and a weekly live call-in Math Chat on local cable TV, both available at the MAA web site at
www.maa.org.

Wake Forest University, Winston-Salem, NC 27109
howards@wfu.edu

Stanford University, Stanford, CA 94305
hutching@math.stanford.edu

Williams College, Williamstown, MA 01267
Frank.Morgan@williams.edu

The infinitude of the primes
Is the subject of plenty of rhymes,
But we can't begin
To prove there's a twin
An infinite number of times.

Contributed by Peter Rosenthal, University of Toronto