We note that $\mathbf{M}$ is an unbounded operator on l_2. For example, using the Euclidean norm the sequence of unit vectors defined by

$$\mathbf{w}_m = \left\{ \{\mathbf{w}_m\}_n = \frac{1}{\sqrt{m}} \quad \text{for } n \le m, \quad \text{and } = 0 \text{ for } n > m \right\}$$

transforms into the sequence $\mathbf{Mw}_m$, which diverges because

$$\|\mathbf{Mw}_m\|^2 = \frac{1}{m} \sum_{n=1}^{m} \left(\sum_{k=1}^{n} \frac{1}{k} \right)^2$$

grows faster than $\ln(m!)/m \approx \ln m$, as m increases.

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More on a Mean Value Theorem Converse

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In a recent MONTHLY article Tong and Braza considered two possible versions of a converse to the Mean Value Theorem [**2**]. For $c \in (a, b)$, a continuous function f on $[a, b]$ that is differentiable on (a, b) satisfies the

1. Weak Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$, and the
2. Strong Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$ with $c \in (\alpha, \beta)$.

In [**2**] the authors give a function that fails the Weak Form (and so fails both forms) at all values in a countable closed set. Borwein and Wang provided a function that fails the Weak Form on a residual set (one whose complement is of first category) that is of Lebesgue measure zero [**1**].

We show that a differentiable function can fail the Weak Form on a set that is both residual and of relative measure arbitrarily close to 1 while the Strong Form must hold on some subset of positive Lebesgue measure. In the rest of this Note *measure* means Lebesgue measure, denoted by λ.

We consider $[a, b] = [0,1]$ and let Z be any measurable set in $[0, 1]$ with $\lambda(Z) < 1$. Let $E \subset [0, 1] \setminus Z$ be an F_σ set with $\lambda(E) = \lambda([0, 1] \setminus Z) > 0$ and E having density 1 at each $x \in E(\lim_{\epsilon \to 0} \lambda(E \cap (x - \epsilon, x + \epsilon))(2\epsilon)^{-1} = 1)$. Let g be an approximately continuous function (at each x the restriction of g to some subset with density 1 at x is continuous at x) such that:

$$\begin{aligned} &1.\ 0 < g(x) \le 1 \quad \text{for } x \in E, \quad \text{and} \\ &2.\ g(x) = 0 \quad \text{for } x \notin E. \end{aligned} \tag{1}$$

A construction of such functions can be found in Zahorski [**3**]. Since g is bounded and approximately continuous it is the derivative of its integral $f(x) = \int_0^x g(t)\,dt$. Therefore $f' \equiv 0$ on Z. We can pick Z to be dense in $[0,1]$ and of measure arbitrarily close to 1 with E having positive measure in every subinterval of $[0,1]$. Then f is strictly increasing and thus has no difference quotient equal to zero. Hence f fails the Weak Form at every point of $\{x | f'(x) = 0\}$ and thus at every point of Z. Since $\{x | f'(x) = 0\}$ is a dense G_δ (it's the complement of the F_σ set E), f fails the Weak Form on a residual set.

However, the following theorem shows that a differentiable function cannot fail the Weak Form almost everywhere.

Theorem 1. *If f is a continuous function on $[a,b]$ that is differentiable on (a,b), then f satisfies the Strong Form on a subset of $[a,b]$ that has positive measure in every subinterval.*

Proof: Let $[\alpha,\beta] \subset [a,b]$. We may assume that f is not linear on any subinterval of $[\alpha,\beta]$ since it would then obviously satisfy the Strong Form there. Let

$$h(x) = \begin{cases} \dfrac{f(x) - f(\alpha)}{x - \alpha} & \text{for } \alpha < x \le \beta \\ f'(\alpha) & \text{for } x = \alpha \end{cases}$$

Then h is continuous on $[\alpha,\beta]$ and $h([\alpha,\beta])$ is some nondegenerate interval $[r,s]$. Since h can have only countably many local extrema we can pick $u \in (\alpha,\beta)$ so that $h(u)$ is not a local extremum. Let c be a point in (α,u) with $f'(c) = h(u)$. Using $p = (c+u)/2$ we see that $f'(c)$ is in the interior of $h([p,\beta])$. Call this interior I. Let g be the restriction of f to the interval $[\alpha,p]$. Then $G = (g')^{-1}(I) \neq \phi$ since it contains c and thus $\lambda(G) > 0$ by the Denjoy-Clarkson Property (the inverse image under a derivative of an open interval is either empty or of positive measure). For each $x \in G$, there is a $y \in [p,\beta]$ with $f'(x) = g'(x) = h(y) = (f(y) - f(\alpha))/(y - \alpha)$. Since $\alpha < x < y$, f satisfies the Strong Form at x. ■

As a final comment, we point out that a differentiable function can fail the Strong Form on a set of positive measure and still satisfy the Weak Form on all of (a,b). As an example we can simply extend our function g in (1) to the interval $[0,4]$ as follows: Let

$$G(x) = \begin{cases} g(x) & 0 \le x \le 1 \\ -g(1)(x-2) & 1 < x \le 2 \\ 0 & 2 < x \le 3 \\ (x-3) & 3 < x \le 4 \end{cases}$$

and set $F(x) = \int_0^x G(t)\,dt$. Then F still fails the Strong Form on the set Z above but satisfies the Weak Form on $(0,4)$. This is because $0 \le G = F' < 1$ on $(0,4)$ while the difference quotients for F inside the interval $(2,4)$ assume all values in $[0,1)$.

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