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Hint: Remember the hexagon inscribed in the conic that we used in the construction of the elements of $\mathbf{C}(0)$.

6. *Assume that triangle ABC has area S and that the radius of its circumscribed circle G is R. We draw a circle K concentric with G and with radius r. From a point P of K we draw its projections U, V, W on the sides of ABC. Determine, as a function of S, R, and r, the area of the triangle UVW.*

Hint: The same as in Exercise 4. Answer: $Area(UVW) = (S/4)(1 - r^2/R^2)$, *having selected the appropriate orientation so that the triangle UVW has positive area when* $r < R$.

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Another Short Proof of Ramanujan's Mod 5 Partition Congruence, and More

Michael D. Hirschhorn

We present another novel short proof of Ramanujan's partition congruence

$$p(5n + 4) \equiv 0 \pmod 5 \tag{1}$$

in addition to that presented by John L. Drost [**2**], and indeed prove rather more.

Ramanujan made the remarkable observation from a table of values of $p(n)$, the number of partitions of n, that $p(5n + 4)$ is divisible by 5. He observed and conjectured much more, and his conjectures turned out in the main to be correct. He gave a simple proof, based upon identities of Euler and Jacobi, of the conjecture (1), and his proof is essentially the one reproduced in Hardy and Wright [**3**] and referred to by Drost. Ramanujan's proof relies on manipulating power series, and considering coefficients modulo 5. It is my intention to give a proof of a similar sort, more transparent than that of Ramanujan, using only the identity of Jacobi. And further, with a little extra work including the use of Jacobi's triple-product identity, we prove remarkable congruences for the partition function due to Atkin and Swinnerton-Dyer.

As is usual, write $(q)_\infty = \prod_{n\geq 1}(1-q^n)$. Then

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q)_\infty}.$$

We begin with Jacobi's identity [**3**, Theorem 357],

$$(q)_\infty^3 = \sum_{n\geq 0} (-1)^n(2n+1)q^{n(n+1)/2}.$$

Each coefficient is congruent modulo 5 to 0, ± 1 or ± 2. Specifically, the coefficient is congruent to 1 when $n \equiv 0$ or 9 (mod 10), -1 when $n \equiv 4$ or 5 (mod 10), $+2$ when $n \equiv 1$ or 8 (mod 10), -2 when $n \equiv 3$ or 6 (mod 10), and 0 when $n \equiv 2$ or 7 (mod 10). Thus we find that, modulo 5,

$$\begin{aligned}(q)_\infty^3 \equiv{}& \sum_{n\geq 0} q^{10n(10n+1)/2} - \sum_{n\geq 0} q^{(10n+4)(10n+5)/2} - \sum_{n\geq 0} q^{(10n+5)(10n+6)/2}\\ &+ \sum_{n\geq 0} q^{(10n+9)(10n+10)/2} + 2\sum_{n\geq 0} q^{(10n+1)(10n+2)/2} - 2\sum_{n\geq 0} q^{(10n+3)(10n+4)/2}\\ &- 2\sum_{n\geq 0} q^{(10n+6)(10n+7)/2} + 2\sum_{n\geq 0} q^{(10n+8)(10n+9)/2}\\ \equiv{}& \sum_{n\geq 0} q^{50n^2+5n} - \sum_{n\geq 0} q^{50n^2+45n+10} - \sum_{n\geq 0} q^{50n^2+55n+15} + \sum_{n\geq 0} q^{50n^2+95n+45}\\ &+ 2\sum_{n\geq 0} q^{50n^2+15n+1} - 2\sum_{n\geq 0} q^{50n^2+35n+6} - 2\sum_{n\geq 0} q^{50n^2+65n+21}\\ &+ 2\sum_{n\geq 0} q^{50n^2+85n+36}.\end{aligned}$$

Observe that in the first four sums the powers of q are congruent to 0 (mod 5) while in the latter four sums the powers of q are congruent to 1 (mod 5). Thus we have

$$(q)_\infty^3 \equiv X + 2qY,$$

where each of X, Y is a series in powers of q^5.

Also

$$\begin{aligned}(q)_\infty^5 &= \prod_{n\geq 1}(1-q^n)^5 = \prod_{n\geq 1}(1 - 5q^n + 10q^{2n} - 10q^{3n} + 5q^{4n} - q^{5n})\\ &\equiv \prod_{n\geq 1}(1-q^{5n}) \equiv (q^5)_\infty.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \frac{1}{(q)_\infty} = \frac{(q)_\infty^9}{(q)_\infty^{10}} = \frac{\left((q)_\infty^3\right)^3}{\left((q)_\infty^5\right)^2} \equiv \frac{\left((q)_\infty^3\right)^3}{(q^5)_\infty^2} \equiv \frac{(X+2qY)^3}{(q^5)_\infty^2}\\ &\equiv \frac{X^3 + 6qX^2Y + 12q^2XY^2 + 8q^3Y^3}{(q^5)_\infty^2}\\ &\equiv \frac{X^3 + qX^2Y + 2q^2XY^2 + 3q^3Y^3}{(q^5)_\infty^2}.\end{aligned}$$

Comparing terms containing powers of q congruent to 4 modulo 5 on both sides, we see that

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} \equiv 0 \pmod 5. \qquad \blacksquare$$

Notice that, at no extra cost, we obtain the congruences

$$\sum_{n\geq 0} p(5n)q^{5n} \equiv X^3/(q^5)_\infty^2,$$

$$\sum_{n\geq 0} p(5n+1)q^{5n+1} \equiv qX^2Y/(q^5)_\infty^2,$$

$$\sum_{n\geq 0} p(5n+2)q^{5n+2} \equiv 2q^2XY^2/(q^5)_\infty^2, \quad \text{and}$$

$$\sum_{n\geq 0} p(5n+3)q^{5n+3} \equiv 3q^3Y^3/(q^5)_\infty^2.$$

It is not hard to show that each of X, Y is an infinite product. Indeed, as we shall see,

$$X = \prod_{n\geq 1}(1-q^{25n-15})(1-q^{25n-10})(1-q^{25n}), \tag{2}$$

$$Y = \prod_{n\geq 1}(1-q^{25n-20})(1-q^{25n-5})(1-q^{25n}). \tag{3}$$

It follows that

$$\sum_{n\geq 0} p(5n)q^n \equiv \prod_{n\geq 1}\frac{(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n})}{(1-q^{5n-4})^2(1-q^{5n-1})^2},$$

$$\sum_{n\geq 0} p(5n+1)q^n \equiv \prod_{n\geq 1}\frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})},$$

$$\sum_{n\geq 0} p(5n+2)q^n \equiv 2\prod_{n\geq 1}\frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})}, \quad \text{and}$$

$$\sum_{n\geq 0} p(5n+3)q^n \equiv 3\prod_{n\geq 1}\frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2}.$$

These remarkable results are due to Atkin and Swinnerton-Dyer [**1**, Theorem 1].

We now show that X, Y are the infinite products claimed in (2) and (3). We have

$$X = \sum_{n\geq 0} q^{50n^2+5n} - \sum_{n\geq 0} q^{50n^2+45n+10} - \sum_{n\geq 0} q^{50n^2+55n+15} + \sum_{n\geq 0} q^{50n^2+95n+45}$$

In the first sum, replace n by $-n$, in the third replace n by $-n-1$, and in the fourth replace n by $n-1$. Then we find

$$\begin{aligned} X &= \sum_{n\leq 0} q^{50n^2-5n} - \sum_{n\geq 0} q^{50n^2+45n+10} - \sum_{n\leq -1} q^{50n^2+45n+10} + \sum_{n\geq 1} q^{50n^2-5n} \\ &= \sum_{n=-\infty}^{\infty} q^{50n^2-5n} - \sum_{n=-\infty}^{\infty} q^{50n^2+45n+10} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2-5n)/2}. \end{aligned}$$

The terms for n even in the final sum correspond to the first sum on the line above; the terms for n odd to the second sum.

In the same way, we find

$$\begin{aligned} Y &= \sum_{n\ge 0} q^{50n^2+15n} - \sum_{n\ge 0} q^{50n^2+35n+5} - \sum_{n\ge 0} q^{50n^2+65n+20} + \sum_{n\ge 0} q^{50n^2+85n+35} \\ &= \sum_{n\le 0} q^{50n^2-15n} - \sum_{n\ge 0} q^{50n^2+35n+5} - \sum_{n\le -1} q^{50n^2+35n+5} + \sum_{n\ge 1} q^{50n^2-15n} \\ &= \sum_{n=-\infty}^{\infty} q^{50n^2-15n} - \sum_{n=-\infty}^{\infty} q^{50n^2+35n+5} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2-15n)/2}. \end{aligned}$$

To complete the proof, we now invoke Jacobi's triple product identity [**3**, Theorem 352], in the form

$$\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{(n^2-n)/2} = \prod_{n\ge 1} (1 - aq^{n-1})(1 - a^{-1}q^n)(1 - q^n)$$

with q replaced by q^{25} and a replaced by q^{10} and q^5, respectively.

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