The TEAM Approach to Investing

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1. DESCRIPTION OF STRATEGIES. We consider an investor with a portfolio of "stock-like" investments (e.g., an S & P 500 Index fund) and "cash-like" investments (e.g., Treasury bills, CDs, and high-grade commercial paper). $S(t)$ and $C(t)$ denote the values at time t of this "stock fund" and "cash fund," respectively, and

$$V(t) = S(t) + C(t) \tag{1.1}$$

denotes the total portfolio value at time t. The ratios $r(t,h) = [V(t+h) - V(t)]/V(t)$ have been studied exhaustively for many such portfolios and for time increments h ranging from days, weeks, and months, to quarters, years, and even longer. For short intervals h, the ratios $r(t,h)$ may be positive or negative, are seldom large, and are difficult to distinguish from independent identically distributed random variables with constant (or slowly varying) means and variances.

At an initial time t_0, we suppose amounts S_0 and C_0 are invested in the stock fund and cash fund, respectively. We compare the results of two strategies. The first is a buy-and-hold strategy with no exchanges between the stock fund and cash fund. The second involves periodically reallocating money between the stock fund and cash fund in a way that maximizes a certain function that is described in the next section. We call the first the BH (Buy-and-Hold) strategy and the second the TEAM (Target Equity Allocation Management) strategy. Our main result (Proposition 1) is that the TEAM strategy produces a higher expected total portfolio value than does the BH strategy for the same level of risk.

What are the reallocations in the TEAM strategy? Pick a convenient time interval h, and let $t_i = t_0 + ih$ for $i = 1, 2, \dots, n$. Suppose the investor moves money back and forth between the stock fund and cash fund only at the instants t_i for $i = 1, 2, \dots, n-1$, in an attempt to obtain a large total portfolio value $V(t_n) = S(t_n) + C(t_n)$. For $i = 1, 2, \dots, n$, let s_i (respectively, c_i) denote the rate of return of the stock fund (respectively, cash fund) during the i-th time period. In the TEAM strategy, money is moved between the stock fund and cash fund at the instants t_i so that

$$S(t_i) = S_0(1 + c_1)(1 + c_2) \cdots (1 + c_i) \tag{1.2}$$

for $i = 1, 2, \dots, n-1$. The reason for these stock fund allocations is explained in the next section. In contrast, in the BH strategy,

$$S(t_i) = S_0(1 + s_1)(1 + s_2) \cdots (1 + s_i) \tag{1.3}$$

for each i.

2. ANALYSIS OF STRATEGIES. We suppose that the s_i's (respectively, c_i's) are independent identically distributed random variables with means μ_s and variances σ_s^2 (respectively, means μ_c and variances σ_c^2). We assume $\mu_s > \mu_c$ and $\sigma_s^2 > \sigma_c^2$. The first inequality reflects the fact that stock-like assets tend to appreciate more rapidly than cash-like assets, while the second inequality expresses the fact that stock-like assets appreciate at a riskier (less predictable) rate than cash-like assets. Generally speaking, investors are paid to bear risk, with extra pay for extra risk.

Let Δ_i denote the dollar amount transferred from the stock fund to the cash fund at time t_i. If $\Delta_i < 0$, money is transferred from the cash fund to the stock fund. (In the BH strategy $\Delta_1 = \cdots = \Delta_{n-1} = 0$.) Write $S_i = S(t_0 + ih)$, $C_i = C(t_0 + ih)$, and $V_i = V(t_0 + ih)$. For any sequence $\Delta_1, \ldots, \Delta_{n-1}$,

$$S_i = S_{i-1}(1 + s_i) - \Delta_i \tag{2.1}$$

and

$$C_i = C_{i-1}(1 + c_i) + \Delta_i \tag{2.2}$$

for $i = 1, \ldots, n$. We may suppose $\Delta_n = 0$ since Δ_n is of no consequence. Then

$$V_i = S_i + C_i = V_{i-1}(1 + c_i) + S_{i-1}(s_i - c_i) \tag{2.3}$$

for each i. Note that $V(t) = S(t) + C(t)$ is continuous across the boundaries (instants) separating consecutive time intervals, even though $S(t)$ and $C(t)$ are not.

The investor may cause the quantities $S_0, S_1, \ldots, S_{n-1}$ to assume any values in the intervals $[0, V_0], [0, V_1], \ldots, [0, V_{n-1}]$ by choosing S_0 and $\Delta_1, \ldots, \Delta_{n-1}$ appropriately. Even these constraints may be eliminated by "selling short" stock-like assets or cash-like assets, e.g., borrowing money in the latter case. Therefore, it seems natural to ignore the constraints $0 \leq S_i \leq V_i$ during the initial search for an effective investment strategy.

Now let $g_0 = 1$ and

$$g_i = g_{i-1}(1 + c_i) = (1 + c_1) \cdots (1 + c_i)$$

for $i \geq 1$. Making the substitutions $V_i = g_i X_i$ and $S_i = g_i u_{i+1}$ in (2.3) gives

$$X_i = X_{i-1} + R_i u_i \tag{2.4}$$

where

$$R_i = (s_i - c_i)/(1 + c_i) \tag{2.5}$$

for each i. Then $X_0 = V_0$ and

$$X_i = X_0 + R_1 u_1 + \cdots + R_i u_i \tag{2.6}$$

for each $i \geq 1$. The investor's choice of the numbers $u_1, \ldots, u_n$ is entirely unrestricted, and the investor presumably chooses $u_1, \ldots, u_n$ in an attempt to maximize an appropriate function of portfolio performance.

For subsequent calculations it is useful to assume that $\sigma_c^2 = \mathrm{Var}(c_i) = 0$ for each i, so that $c_1 = \cdots = c_n = \mu_c = c$, a numerical constant, and then $g_i = (1 + c)^i$ for each i. In practice, $\mathrm{Var}(c_i)$ is usually so much smaller than $\mathrm{Var}(s_i)$ that it seems natural to ignore $\mathrm{Var}(c_i)$. Indeed, the availability of a "riskless rate c of return" is an essential feature of the "Black-Scholes environment" in which most financial analysis is performed.

Next let $M_n = V_0 g_n$, which is the amount the cash fund would contain if the entire initial investment V_0 had been placed in the cash fund and left alone. We would like the terminal value V_n of our chosen portfolio of stocks and cash to exceed M_n by a large amount. However, to take into account the riskiness of our portfolio, we choose to maximize the function

$$f(V_n - M_n) = E(V_n - M_n)/\sqrt{\mathrm{Var}(V_n - M_n)}\,, \tag{2.7}$$

which increases with the expected value of $V_n - M_n$ and decreases with the standard deviation of $V_n - M_n$. This function is essentially the Sharpe reward-to-variability ratio [2]. Now from (2.6)

$$V_n - M_n = g_n(X_n - X_0) = g_n(R_1u_1 + \cdots + R_nu_n)$$
$$= (1+c)^n(R_1u_1 + \cdots + R_nu_n) \tag{2.8}$$

under our assumption that $c_i = c$ for each i. Then (2.5) and our assumption that the s_i's are independent identically distributed random variables imply that the R_i's are independent identically distributed random variables with means $(\mu_s - c)/(1+c)$ and variances $\sigma_s^2/(1+c)^2$.

We now consider a special case for the u_i's; namely, we assume that the u_i's are chosen in a way that does not depend on the stock fund rates of return $s_1, s_2, \dots, s_n$. Then (2.7) and (2.8) imply

$$f(V_n - M_n) = K\langle \mathbf{e}, \mathbf{u}\rangle / \|\mathbf{u}\|_2 \le K\|\mathbf{e}\|_2 = K\sqrt{n} \tag{2.9}$$

where $\mathbf{u}$ is the vector $(u_1, \dots, u_n)$, $\mathbf{e}$ is the vector with n components each equal to 1, and $K = (\mu_s - c)/\sigma_s$. The inequality in (2.9) is a consequence of the Cauchy-Schwarz inequality, with equality if and only if $\mathbf{u} = \lambda\mathbf{e}$ for some positive scalar λ. Since $u_1 = g_0u_1 = S_0$, then equality in (2.9) occurs precisely when $\mathbf{u} = S_0\mathbf{e}$ with S_0 positive. Then

$$S_i = g_iu_{i+1} = S_0(1+c)^i \tag{2.10}$$

for $i = 1, \dots, n-1$. Note that (2.10) is the same as (1.2) under the assumption that $c_i = c$ for each i. This optimal strategy corresponds to the choices $\Delta_i = (s_i - c)S_{i-1}$ for $i = 1, \dots, n-1$ in (2.1). Each $S_i/S_{i-1} = 1 + c$, which means that the stock fund allocations increase at the riskless rate c. This strategy is called the Target Equity Allocation Management (TEAM) strategy since it allocates resources between the riskier stock fund and more conservative cash fund in a way calculated to achieve the modest equity targets $S_i = S_0(1+c)^i$ in the stock fund, while transferring anticipated surpluses to the cash fund.

Although the TEAM strategy maximizes the function (2.7) among all strategies for which the u_i's do not depend on the stock fund rates of return $s_1, \dots, s_n$, we should expect some feedback type strategies (in which the u_i's depend on the s_j's) to produce greater values for $f(V_n - M_n)$. In the buy-and-hold (BH) strategy, the u_i's do depend on the s_j's since $\Delta_1 = \cdots = \Delta_{n-1} = 0$ imply that

$$(1+c)^iu_{i+1} = S_i = S_0(1+s_1)\cdots(1+s_i) \tag{2.11}$$

for $i = 1, \dots, n-1$. Hence there is still some work to do to show that the TEAM strategy imparts a higher value to the function $f(V_n - M_n)$ than does the BH strategy.

To avoid confusion with other strategies, let B_n rather than V_n denote the terminal portfolio value for the BH strategy. From (2.5) and (2.11), R_i depends on s_i, whereas u_i depends on $s_1, \dots, s_{i-1}$. Thus, R_i and u_i are independent, and (2.8) implies

$$E(B_n - M_n) = (1+c)^n\left(\bar{R}_1\bar{u}_1 + \cdots + \bar{R}_n\bar{u}_n\right) \tag{2.12}$$

where

$$\bar{R}_i = E(R_i) = (\mu_s - c)/(1+c) \tag{2.13}$$

$$\bar{u}_i = E(u_i) = S_0(1+\mu_s)^{i-1}/(1+c)^{i-1} \tag{2.14}$$

for $i = 1, \ldots, n$. A straightforward but lengthy calculation shows that

$$\operatorname{Var}(B_n - M_n) > (1+c)^{2(n-1)} \sigma_s^2 \left(\bar{u}_1^2 + \cdots + \bar{u}_n^2 \right). \tag{2.15}$$

Then (2.7), (2.12), (2.13), and (2.15) imply

$$f(B_n - M_n) < K \langle \mathbf{e}, \mathbf{w} \rangle / \|\mathbf{w}\|_2 < K \|\mathbf{e}\|_2 = K\sqrt{n} \tag{2.16}$$

where $\mathbf{w} = (\bar{u}_1, \ldots, \bar{u}_n)$, $\mathbf{e} = (1, \ldots, 1)$, and $K = (\mu_s - c)/\sigma_s$. The second inequality in (2.16) is a consequence of the Cauchy-Schwarz inequality and the fact that $\mathbf{w} \neq \lambda \mathbf{e}$ for a scalar λ since $\mu_s > c$ in (2.14).

Now if T_n is the terminal portfolio value for the TEAM strategy, then

$$f(B_n - M_n) < K\sqrt{n} = f(T_n - M_n). \tag{2.17}$$

For the TEAM strategy, the analogs of (2.12) and (2.15) are

$$E(T_n - M_n) = (1+c)^n \left(\overline{R}_1 S_0 + \cdots + \overline{R}_n S_0 \right) = (1+c)^{n-1} n (\mu_s - c) S_0 \tag{2.18}$$

$$\operatorname{Var}(T_n - M_n) = (1+c)^{2(n-1)} n \sigma_s^2 S_0^2. \tag{2.19}$$

Since $\mu_s > c$, then (2.12), (2.14), (2.15), (2.18), and (2.19) imply that for the same initial allocations in the BH strategy and in the TEAM strategy, the BH strategy has a higher expected terminal portfolio value and a higher variance than does the TEAM strategy. Next, observe that the value $f(T_n - M_n) = K\sqrt{n}$ in (2.17) does not depend on the initial allocations in the stock fund and cash fund. Hence we could increase the initial allocation to an amount S_0' in the stock fund in the TEAM strategy (while correspondingly decreasing the initial allocation to an amount C_0' in the cash fund) until $\operatorname{Var}(T_n' - M_n) = \operatorname{Var}(B_n - M_n)$, where T_n' is the TEAM terminal portfolio value for initial investments of S_0' and C_0' in the stock and cash funds, respectively. Then (2.7) and (2.17) imply $E(T_n') > E(B_n)$.

We now list the primary assumptions that are used in our derivation and then state the proposition we have proved.

The cash fund rate of return is constant over all time periods. (2.20)

The stock fund rates of return in each time period are independent identically distributed random variables. (2.21)

The expected stock fund rate of return exceeds the cash fund rate of return. (2.22)

There are no taxes or transaction costs. (2.23)

Proposition 1. *Suppose an amount S_0 (respectively, S_0') is invested initially in the stock fund in the BH strategy (respectively, TEAM strategy) and an amount C_0 (respectively, C_0') is invested initially in the cash fund in the BH strategy (respectively, TEAM strategy). Let B_n (respectively, T_n') denote the terminal portfolio value for the BH (respectively, TEAM) strategy for these initial allocations. Suppose S_0' and C_0' are chosen so that $(S_0' + C_0') = (S_0 + C_0)$ and so that the standard deviations of T_n' and B_n are equal. Then under the assumptions* (2.20) *through* (2.23), *the expected value of T_n' is greater than that of B_n. Hence for the same level of risk (as measured by standard deviation of terminal portfolio value), the TEAM strategy produces a higher expected terminal portfolio value than does the BH strategy.*

If S_0 is large relative to C_0, then it could happen that $S_0' > (S_0 + C_0)$, in which case $C_0' < 0$. This corresponds to borrowing money in the TEAM strategy. Also, reallocations in the TEAM strategy could require borrowing money.

The assumptions (2.20) through (2.23) are not precisely satisfied in practice, and our model is a very simplified financial model. Nevertheless it is interesting to compare simulation results using the BH and TEAM strategies.

3. SIMULATION RESULTS. We examine the results of some simulations using historical data from 1926 to 1995. Most of the data comes from Ibbotson and Sinquefield [**1**, pp. 54–55]. For rates of return for our stock fund, we use rates of return for Common Stocks (which correspond to the S & P 500 Index); for our cash fund, we use the rates of return for U.S. Treasury Bills. We exclude transaction costs and taxes, and consider 14 non-overlapping 5-year periods from 1926 to 1995. In each 5-year period, the TEAM strategy reallocations occur annually, so $n = 5$. There are no reallocations in the BH strategy during each 5-year period. The data in [**1**] show that the serial correlation for annual rates of return is near zero for a stock fund such as an S & P 500 Index fund, which is consistent with what we would expect from (2.21).

Since the TEAM strategy involves lower risk than the BH strategy when the BH and TEAM strategies have the same initial allocations, one can increase the initial percentage allocation to the stock fund when using the TEAM strategy. Let S_0 (respectively, S_0') denote the initial percentage allocation in the stock fund when using the BH strategy (respectively, TEAM strategy), and set

$$a = S_0'/S_0. \tag{3.1}$$

The TEAM simulation results in Table 1 correspond to $a = 1.4$. We also performed simulations with $a = 1.0$, 1.1, 1.2, 1.3, 1.5, 1.6, 1.7, and 1.8. As one would expect, the average terminal value for the TEAM strategy (and its standard deviation) increased as a increased. However the standard deviation for the TEAM strategy was less than the standard deviation for the BH strategy for $a \leq 1.6$. For $1.2 \leq a \leq 1.6$, the TEAM strategy produced a higher terminal value than the BH strategy in at least 12 of the 14 five-year periods.

TABLE 1 Total Returns for Two Strategies

Strategy	Buy and Hold (BH)		TEAM
Initial Value	$1.0000		$1.0000
Stock fraction	.5000		.7000
Cash fraction	.5000		.3000
Time Periods	Terminal Values ($)		Difference (TEAM − BH)
1926–1930	1.3476	1.5202	.1726
1931–1935	1.0964	1.3220	.2256
1936–1940	1.0150	1.1413	.1263
1941–1945	1.6010	1.6461	.0451
1946–1950	1.3221	1.3977	.0756
1951–1955	1.9973	1.9555	−.0418
1956–1960	1.3336	1.4382	.1046
1961–1965	1.5135	1.5951	.0816
1966–1970	1.2413	1.2481	.0068
1971–1975	1.2477	1.3398	.0921
1976–1980	1.6874	1.7785	.0911
1981–1985	1.8094	1.9154	.1060
1986–1990	1.6226	1.7063	.0837
1991–1995	1.6933	1.7713	.0780
Average	1.4663	1.5554	.0891
Standard Deviation	.2823	.2520	.0643

For $a = 1.4$ in Table 1, the average difference of .0891, i.e., an average 5-year total return difference of 8.91% for the TEAM strategy over the BH strategy, is significantly greater than zero at the 1% level for the t-statistic for the variable (TEAM − BH) terminal value. One might argue that this variable is not normally distributed, and hence the t-statistic might not be appropriate. Further statistical justification for concluding that the (TEAM − BH) terminal value is significantly greater than zero is the fact that the TEAM terminal value exceeded the BH terminal value in 13 of the 14 time periods. Hence with an appropriate choice of a, the TEAM strategy can produce a higher terminal value than the BH strategy, with no greater risk than the BH strategy.

The data in Table 1 correspond to the assumption that each stock fund investment S_i' in the TEAM strategy is limited to the total value T_i'. If borrowing is allowed, the TEAM strategy results are slightly better.

REFERENCES

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From the MONTHLY 25 Years Ago...

THE DERIVATIVE SONG

Words by Tom Lehrer—Tune: "There'll be Some Changes Made"

You take a function of x and you call it y,
Take any x-nought that you care to try,
You make a little change and call it delta x,
The corresponding change in y is what you find nex',
And then you take the quotient and now carefully
Send delta x to zero, and I think you'll see
That what the limit gives us, if our work all checks,
Is what we call dy/dx,
It's just dy/dx

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