NOTES

Edited by **Jimmie D. Lawson and William Adkins**

A Physically Motivated Further Note on the Mean Value Theorem for Integrals

William J. Schwind, Jun Ji, and Daniel E. Koditschek

The purpose of this Note is to extend the following result by Zhang concerning the mean value theorem for integrals [**4**], which in turn was an extension of Jacobson's result [**1**].

Theorem 1. *Suppose the function f is continuous on the interval* $[a, b]$, *and is k times differentiable at a with* $f^{(i)}(a) = 0$ $(i = 1, 2, \ldots, k-1)$, $f^{(k)}(a) \neq 0$. *If* ξ_x *is such that*

$$\int_a^x f(t)\,dt = f(\xi_x)(x-a), \tag{1}$$

then

$$\lim_{x \to a} \frac{\xi_x - a}{x - a} = \frac{1}{(k+1)^{\frac{1}{k}}}. \tag{2}$$

As a side note, Jacobson [**1**] states that his result may fall into the category of "interesting facts we once knew, but have now forgotten." This indeed seems to be the case, as the right-hand side of (2) is derived in [**2**, p. 78].

We extend Theorem 1 to a considerably larger function class, which includes functions such as $f(t) = \sqrt{t-a}$ and $f(t) = 1/\sqrt{t-a}$ that do not satisfy the hypotheses of Theorem 1. Additionally, we apply the new results to obtain approximations of integrals appearing in familiar engineering settings. In fact our consideration of this problem was motivated by the desire to approximate, in closed form, integrals arising from a class of central force problems and to do so independently of the particular mathematical form of the force law itself.

Theorem 2. *Suppose f is continuous on* $(a, b]$ *and g is integrable on* (a, b) *with* $g(t) \geq 0$ *for all* $t \in (a, b)$. *Let* $x \in (a, b]$. *If both* $\lim_{t \to a}(f(t) - K)/(t-a)^r$ *and* $\lim_{t \to a} g(t)/(t-a)^s$ *exist and are nonzero for some constant K, some nonzero r, and some* $s > -1$ *with* $r + s > -1$, *then*

1. *there exists a* $\xi_x \in (a, x]$ *such that*

$$\int_a^x f(t) g(t)\,dt = f(\xi_x) \int_a^x g(t)\,dt; \tag{3}$$

2. *for any such choice of* ξ_x,

$$\lim_{x \to a} \frac{\xi_x - a}{x - a} = \left(\frac{s + 1}{r + s + 1} \right)^{\frac{1}{r}}. \tag{4}$$

Proof: Define $C_1 = \lim_{t \to a}(f(t) - K)/(t - a)^r$, $C_2 = \lim_{t \to a} g(t)/(t - a)^s$, $\epsilon_1(t) = (f(t) - K)/(t - a)^r - C_1$, and $\epsilon_2(t) = g(t)/(t - a)^s - C_2$, so that

$$f(t) = K + C_1(t - a)^r + \epsilon_1(t)(t - a)^r \tag{5}$$

and

$$g(t) = C_2(t - a)^s + \epsilon_2(t)(t - a)^s, \tag{6}$$

where $\epsilon_i(t) \to 0$ as $t \to a$ for $i = 1,2$.

First we must justify that the integral on the left hand side of (3) actually exists. Fix $\delta \in (a, b)$ such that $\epsilon_i(t) \leq 1$ for $a < t < \delta$. Now $|f(t)g(t)| \leq K_1(t - a)^s + K_2(t - a)^{s+r}$ for some constants K_1 and K_2. The Dominated Convergence Theorem [**3**, p. 291] ensures that $\int_a^\delta f(t)g(t)\,dt$ exists. Therefore, $\int_a^x f(t)g(t)\,dt$ exists.

Proof of (1). Case 1 ($r > 0$). It is easily seen from (5) that $\lim_{t \to a} f(t) = K$. Define $F(a) = K$ and $F(t) = f(t)$ for $t \in (a, b]$. Clearly, $F(t)$ is continuous on $[a, b]$. By applying the Integral Mean Value Theorem [**3**, p. 281] to $F(t)$, there exists a ξ_x in (a, x) such that

$$\int_a^x f(t)g(t)\,dt = \int_a^x F(t)g(t)\,dt = F(\xi_x)\int_a^x g(t)\,dt = f(\xi_x)\int_a^x g(t)\,dt.$$

Case 2 ($r < 0$). Without loss of generality take $C_1 > 0$ and $\int_a^x g(t)\,dt > 0$. Then (5) requires that $\lim_{x \to a} f(t) = +\infty$. Thus there exists a positive δ such that

$$f(t) > \frac{\int_a^x f(t)g(t)\,dt}{\int_a^x g(t)dt} \equiv \eta, \quad \text{for all } a < t < a + \delta.$$

Therefore, a ξ_x satisfying (3), if it exists, is not contained in $(a, a + \delta)$. For $a_1 \in (a, a + \delta)$, f is continuous on $[a_1, x]$ and $f(t) > \eta$ on $(a, a_1]$. Hence,

$$\min_{t \in (a, x]} f(t) = \min_{t \in [a_1, x]} f(t).$$

Therefore,

$$\max_{t \in [a_1, x]} f(t) \geq f(a_1) > \eta \geq \min_{t \in (a, x]} f(t) = \min_{t \in [a_1, x]} f(t).$$

By applying the Intermediate Value Theorem to $f(t)$ on $[a_1, x]$, we obtain $\xi_x \in [a_1, x] \subseteq (a, x]$ such that $f(\xi_x) = \eta$.

Proof of (2). Substituting (5) and (6) into the left hand side of (3), one sees easily that

$$\begin{aligned} \int_a^x f(t)g(t)\,dt = {} & \frac{C_2 K(x - a)^{s+1}}{s + 1} + \frac{C_1 C_2(x - a)^{r+s+1}}{r + s + 1} \\ & + C_2\int_a^x \epsilon_1(t)(t - a)^{r+s}\,dt + K\int_a^x \epsilon_2(t)(t - a)^s\,dt \\ & + C_1\int_a^x \epsilon_2(t)(t - a)^{r+s}\,dt + \int_a^x \epsilon_1(t)\epsilon_2(t)(t - a)^{r+s}\,dt. \end{aligned} \tag{7}$$

On the other hand, substituting (5) and (6) into the right hand side of (3), we find

$$
\begin{aligned}
f(\xi_x)\int_a^x g(t)\,dt = {} & \frac{C_2 K (x-a)^{s+1}}{s+1} + \frac{C_1 C_2 (\xi_x - a)^r (x-a)^{s+1}}{s+1} \\
& + \frac{C_2 \epsilon_1(\xi_x)(\xi_x - a)^r (x-a)^{s+1}}{s+1} + K\int_a^x \epsilon_2(t)(t-a)^s\,dt \\
& + C_1(\xi_x - a)^r \int_a^x \epsilon_2(t)(t-a)^s\,dt \\
& + \epsilon_1(\xi_x)(\xi_x - a)^r \int_a^x \epsilon_2(t)(t-a)^s\,dt. \qquad (8)
\end{aligned}
$$

Equating (7) and (8) and simplifying gives

$$
U\frac{(\xi_x - a)^r}{(x-a)^r} = V, \qquad (9)
$$

where

$$
\begin{aligned}
U = 1 + \frac{\epsilon_1(\xi_x)}{C_1} & + \frac{(s+1)\epsilon_1(\xi_x)\int_a^x \epsilon_2(t)(t-a)^s\,dt}{C_1 C_2 (x-a)^{s+1}} \\
& + \frac{(s+1)\int_a^x \epsilon_2(t)(t-a)^s\,dt}{C_2(x-a)^{s+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
V = \frac{s+1}{r+s+1} & + \frac{(s+1)\int_a^x \epsilon_1(t)(t-a)^{r+s}\,dt}{C_1(x-a)^{r+s+1}} \\
& + \frac{(s+1)\int_a^x \epsilon_2(t)(t-a)^{r+s}\,dt}{C_2(x-a)^{r+s+1}} + \frac{(s+1)\int_a^x \epsilon_1(t)\epsilon_2(t)(t-a)^{r+s}\,dt}{C_1 C_2 (x-a)^{r+s+1}}.
\end{aligned}
$$

Because $|\xi_x - a| \le |x-a|$ and

$$
\lim_{x\to a} \frac{\int_a^x d(t)(t-a)^m\,dt}{(x-a)^{m+1}} = 0 \quad \text{if} \quad \lim_{x\to a} d(t) = 0, \quad m > -1,
$$

we obtain

$$
\lim_{x\to a} U = 1 \quad \text{and} \quad \lim_{x\to a} V = \frac{s+1}{r+s+1} > 0. \qquad (10)
$$

Now (4) follows from (9) and (10). ∎

Often, a good choice for K in Theorem 2 is to take $K = \lim_{t\to a} f(t)$ if the limit exists or $K = 0$ otherwise.

A few immediate observations can be made when $g(t) \equiv 1$. In this case (3) is identical to (1), the statement of the Mean Value Theorem used by Jacobson and Zhang.

Observation 1. *If* $g(t) \equiv 1$ *in Theorem* 2, *then* $s = 0$ *and*

$$\lim_{x \to a} \frac{\xi_x - a}{x - a} = \left(\frac{1}{r + 1} \right)^{\frac{1}{r}} \tag{11}$$

for some nonzero $r > -1$.

Notice that the form of the limiting value in (11) is identical to that in (2), but $r \in (-1, \infty) \setminus \{0\}$, while k is only a natural number.

Observation 2. *Theorem* 1 *is a special case of Theorem* 2.

To see this, assume the hypothesis in Theorem 1; then

$$\lim_{t \to a} \frac{f(t) - f(a)}{(t - a)^k} = \lim_{t \to a} \frac{f^{(k-1)}(t)}{k!(t - a)} = \lim_{t \to a} \frac{f^{(k-1)}(t) - f^{(k-1)}(a)}{k!(t - a)} = \frac{f^{(k)}(a)}{k!} \neq 0,$$

where the first equality is obtained by using L'Hospital's rule $k - 1$ times while the the third one follows from the definition of $f^{(k)}(a)$.

Observation 3. *If, motivated by* (4), *we approximate* ξ_x *by*

$$\hat{\xi}_x = a + \left(\frac{s + 1}{r + s + 1} \right)^{\frac{1}{r}} (x - a) \quad \text{for } x \text{ near } a,$$

and replace ξ_x *by* $\hat{\xi}_x$ *in* (3), *we obtain an approximation scheme to the integral*

$$\int_a^x f(t) g(t)\, dt \approx f(\hat{\xi}_x) \int_a^x g(t)\, dt \quad \text{for } x \text{ near } a. \tag{12}$$

A CENTRAL FORCE EXAMPLE. Consider the simple central force problem in which a mass on a spring is restricted to move in the vertical direction. Let the spring potential be given by $U(y)$, where y is the distance from the ground to the mass. Then the dynamics are given by

$$\ddot{y} = -g - \frac{DU(y)}{m}, \tag{13}$$

where g is the acceleration due to gravity and $-DU(y)$ is the spring force; as a matter of notation, $DU(y) = U'(y)$.

Since we assume no losses, the total energy is a constant of motion and we can formulate the integral for time as

$$T(y) = \frac{1}{\sqrt{2g}} \int_{y_i}^{y} \frac{d\psi}{\sqrt{(y_b - \psi) + \frac{1}{mg}(U(y_b) - U(\psi))}}, \tag{14}$$

where y_b is location of the mass when the vertical velocity is zero.

Under reasonable assumptions on the spring potential, which will be made clear in the following, we can approximate (14) using Observation 3.

Other problems, having integrals that resemble (14), could also be considered. Examples are the integrals for swing time and swing angle of a rotating mass on a spring or the integral for swing time of a simple pendulum. It is well known that the solution of this latter example can be formulated as an elliptic integral and thus several well-known approximation techniques can be applied. The results presented in this Note could be used as well, but additionally, and more importantly, they may also be applied if there are other forces, such as a torsional spring, acting on the pendulum.

We focus on applying the results of Theorem 2 to our central force problem. Suppose we desire to approximate the length of time it takes for the mass to move from the point of maximal compression, y_b, to some other location, y. The result is given by (14) with $y_i = y_b$. Furthermore, suppose we wish to solve this problem without assuming a particular functional form for the spring potential, $U(y)$. In such a case, it is impossible to integrate (14) in closed form. However, (12) provides an approximation to the integral of interest.

To apply Theorem 2, we need to factor the integrand of (14) into the product of two functions, f and g. This factorization is by no means unique and each choice results in a slightly different approximation to the integral. We illustrate two possible choices and discuss the advantages and disadvantages of each.

Let us first consider

$$g_1(y) = 1 \quad \text{and} \quad f_1(y) = \frac{1}{\sqrt{(y_b - y) + \frac{1}{mg}(U(y_b) - U(y))}}.$$

In this case $s_1 = 0$. Choosing $K = 0$ and under reasonable assumptions on $U(y)$, that is,

$$\lim_{y \to y_b} \frac{U(y) - U(y_b)}{y - y_b} = U'(y_b) \quad \text{exists and is not equal to } -mg, \tag{15}$$

we find $r_1 = -1/2$. Note that the exception $U'(y_b) = -mg$, implies $\ddot{y}_b = 0$. Since by definition $\dot{y}_b = 0$, this corresponds to an equilibrium of the system. From (12), we have an approximation to the integral (14)

$$\hat{T}_1(y) = \frac{y - y_b}{\sqrt{2g}\sqrt{\left(y_b - \hat{\xi}_y^1\right) + \frac{1}{mg}\left(U(y_b) - U\left(\hat{\xi}_y^1\right)\right)}}, \tag{16}$$

where $\hat{\xi}_y^1 = y_b + (1/4)(y - y_b)$.

Now consider

$$g_2(y) = \frac{1}{\sqrt{y - y_b}} \quad \text{and} \quad f_2(y) = \frac{1}{\sqrt{-1 + \frac{1}{mg}\left(\frac{U(y_b) - U(y)}{y - y_b}\right)}}.$$

Here, we find $s_2 = -1/2$. If we assume (15), we can choose $K = \lim_{y \to y_b} f_2(y)$. In order to solve for r_2, we need to apply L'Hospital's theorem. This, however, does not provide enough information to determine r_2—we need to know more about the structure of $U(y)$. If, for example, we assume

$$\lim_{y \to y_b} \frac{U'(y)(y - y_b) - (U(y) - U(y_b))}{(y - y_b)^2} = U''(y_b) \neq 0 \quad \text{exists}, \tag{17}$$

we find $r_2 = 1$. If, however, $U''(y_b) = 0$, we must apply L'Hospital again and we find that if

$$\lim_{y \to y_b} \frac{U''(y)(y - y_b)^2 - (U'(y)(y - y_b) - (U(y) - U(y_b)))}{(y - y_b)^3}$$
$$= U'''(y_b) \neq 0 \quad \text{exists},$$

then $r_2 = 2$.

Let us assume (17) holds; this implies $r_2 = 1$. In this case, using (12), we have another approximation to the integral (14),

$$\hat{T}_2(y) = \frac{1}{\sqrt{2g}\sqrt{-1 + \frac{1}{mg}\left(\frac{U(y_b) - U(\hat{\xi}_y^2)}{\hat{\xi}_y^2 - y_b}\right)}} \int_{y_b}^{y} \frac{1}{\sqrt{\psi - y_b}} d\psi$$

$$= \frac{2\sqrt{y - y_b}}{\sqrt{2g}\sqrt{-1 + \frac{1}{mg}\left(\frac{U(y_b) - U(\hat{\xi}_y^2)}{\hat{\xi}_y^2 - y_b}\right)}}, \qquad (18)$$

where $\hat{\xi}_y^2 = y_b + (1/3)(y - y_b)$.

In this approximation strategy, each approach has its own advantages and disadvantages. The second approach has the advantage of extracting the "dominant" behavior of g and integrating that exactly, but has the drawback of requiring more explicit knowledge of the spring potential law in order to calculate r. The first approach, while not offering the exactness of the second, allows greater flexibility because its approximation is based only upon the "uninformed" factorization ($g(t) \equiv 1$), which allows r to be determined with only minimal knowledge of the spring potential law. Therefore, one's choice of approximation depends on the application of interest.

In this Note we provide two different approximations to (14). From Observation 3, we know that these approximations are good for y close to y_b; however, our proof shows that these approximations may be suitable over larger intervals.

ACKNOWLEDGMENTS. We thank Professors Charles Doering and Charles Kicey for their helpful suggestions regarding this paper. This work was supported in part by National Science Foundation Grant IRI – 9612357.

REFERENCES

1. Bernard Jacobson, On the Mean Value Theorem for Integrals, *Amer. Math. Monthly* **89** (1982) 300–301.
2. Ganesh Prasad, *Six Lectures on the Mean Value Theorem of the Differential Calculus*, University of Calcutta, Calcutta, 1931.
3. Karl R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Inc., Belmont, CA, 1981.
4. Zhang Bao-lin, A Note on the Mean Value Theorem for Integrals, *Amer. Math. Monthly* **104** (1997) 561–562.

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109
{schwindw, kod}@eecs.umich.edu

Department of Mathematics and Computer Science, Valdosta State University, Valdosta, GA 31698
junji@valdosta.edu