The Hopping Hoop Revisited

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In this MONTHLY, T. Tokieda [1] recently provided a novel discussion of the hopping hoop problem described by Littlewood [2]. Unfortunately, the analysis in [1] is incorrect. We show that if the "no slipping" condition imposed in [1] and [2] is strictly adhered to, the hoop never becomes airborne. Essential for the observed hop of the hoop is a phase in which there is slippage at the point of contact of the hoop with the supporting surface. In order to expose this and other subtleties not considered in [1], we treat a somewhat more general problem. The view of Littlewood's hopping hoop problem presented here is consciously different from that given in [1]. In offering this alternative way of thinking about Littlewood's hopping hoop, we hope to deepen the reader's intuitive understanding of precisely why the hoop hops.

We consider here a rigid circular hoop of radius R and mass $(1-\lambda)M$ that, at least initially, rolls without slipping along a flat surface. An additional object of mass λM is rigidly attached to the rim of the hoop, with the point of attachment coinciding with the centroid of the object. Here, M is the total mass of the combined system consisting of the hoop and the attached object. The parameter λ is the ratio of the object mass to the total; in the limit $\lambda \to 1$, the problem reduces to the case of the massless hoop considered by Tokieda and Littlewood. The center of mass of the combined system, shown as a dot in Figure 1, lies a distance

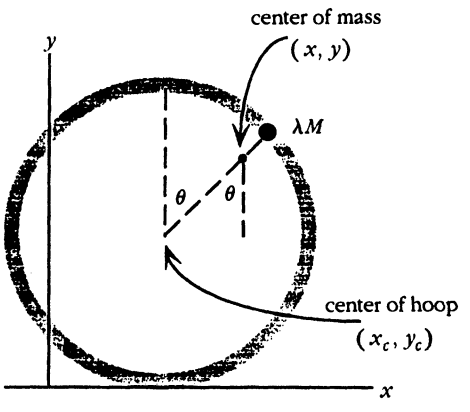

Figure 1. The hopping hoop: a circular hoop of mass $(1-\lambda)M$ with an attached object of mass λM.

λR from the center of the hoop and is situated on the line segment connecting the center of the hoop with the attached object. Let $\theta(t)$ be the angle which this line segment makes with the vertical and let $x(t)$ and $y(t)$ be, respectively, the horizontal and vertical coordinates of the center of mass of the system.

The three coordinates $x(t)$, $y(t)$, and $\theta(t)$ are not independent, but are subject to various constraints:

1. Since the hoop is assumed to be a rigid object that suffers no deformation as it rolls, it is always true that $(x(t)-x_c(t))^2+(y(t)-y_c(t))^2=\lambda^2R^2$, or

equivalently,

$$x(t) - x_c(t) = R\lambda \sin\theta(t) \tag{1}$$

$$y(t) - y_c(t) = R\lambda \cos\theta(t) \tag{2}$$

where $x_c(t)$ and $y_c(t)$ denote, respectively, the horizontal and vertical coordinates of the center of the hoop.

2. Since the supporting surface is also assumed to be rigid, the vertical height of the center of the hoop must satisfy $y_c(t) = R$ so long as the hoop and the surface are in contact.
3. As long as there is no slipping, the horizontal position of the center of the hoop and the angular coordinate $\theta(t)$ are related by $x_c(t) = R\theta(t)$, where we have tacitly set $x_c(0) = R\theta(0) = 0$.

Taken together, these three constraints imply that as long as the hoop rolls without slipping along the supporting surface, the center of mass of the system moves along a curtate (shortened) cycloid, also known as a trochoid.

Every constraint is maintained by a corresponding force. In particular, constraints 2 and 3 result from the force of contact between the hoop and the supporting surface: the condition $y_c(t) = R$ is maintained by the so-called normal force, i.e., the component of the contact force acting perpendicular to the surface, while the "no slipping" constraint, $x_c(t) = R\theta(t)$, is maintained by friction, i.e., by the component of the contact force acting parallel to the surface. Constraint 1, the "rigid hoop" constraint, is maintained by forces within the hoop itself; because these are internal to the system under consideration, they need not concern us here.

All forces external to the system, including those responsible for maintaining the constraints just enumerated, appear in the equations of motion of the center of mass of the system, which are obtained from Newton's Second Law:

$$\ddot{x}(t) = f(t) \tag{3}$$

$$\ddot{y}(t) = n(t) - g \tag{4}$$

Here, g, $n(t)$, and $f(t)$ represent, respectively, the force per unit mass (acceleration) due to gravity, to the normal component of the contact force, and to friction. In order to simplify the equations, we divide all forces by M, the total mass of the system. For brevity we continue to refer to the resulting quantities as "forces" even though they are actually accelerations, i.e., forces per unit mass. When the hoop loses contact with the supporting surface, $n(t) = f(t) = 0$, and the equations (3) and (4) describe a parabolic trajectory.

The most general motion of a rigid body consists of a translation of the center of mass, combined with a rotation about an axis containing the center of mass. In the present case, this rotation is described by the angular variable $\theta(t)$, whose time evolution is governed by the following equation, obtained by considering the torques about the axis perpendicular to the plane of Figure 1 and passing through the center of mass of the system:

$$\frac{I}{M}\ddot{\theta}(t) = n(t)\left[x(t) - x_c(t)\right] - f(t)y(t) \tag{5}$$

In this expression, I is the moment of inertia of the system (hoop + object) about the axis through the system center of mass. It is given by $I = MR^2\{1 - \lambda^2 + \lambda\epsilon\}$, where the last term in brackets requires some comment. Unless the attached object is a point mass (a mathematical idealization), it has a non-vanishing moment of inertia about any axis through its centroid. That "internal" moment of inertia is proportional to the mass λM of the attached object and to the square of some length parameter related to the dimensions of the object in the plane perpendicular to the axis of rotation; the moment of inertia is of the form $\lambda M b^2$. For example, for a cylinder of uniform mass density (a battery!), attached to the hoop so that its longitudinal axis is aligned with the hoop, $b = L/\sqrt{12}$, where L is the length of the cylinder; for the same cylinder, attached to the hoop in such a way that its longitudinal axis is perpendicular to the plane of the hoop, $b = r/\sqrt{12}$, where r is the cylinder radius; and so on. For a point mass, $b = 0$. Instead of b, however, we choose to work with the dimensionless parameter $\epsilon = (b/R)^2$. Since $\epsilon \ll 1$ in almost all cases, one might wonder why we make a large fuss over a quantity that, in practice, is negligible small. The answer is that we wish to examine the limit $\lambda \to 1$, in which the mass of the hoop is negligible relative to that of the attached object. In this limit, the center of mass of the system coincides with the centroid of the object, and the system's angular momentum relative to its center of mass resides entirely in the attached object spinning about its centroid; this angular momentum is zero if the corresponding moment of inertia vanishes, as is the case if we set to zero both the mass of the hoop and the "internal" rotational inertia of the attached object. Alternately, we observe that (5) makes sense only so long as $I \neq 0$, and, in the case of a massless hoop, I is nonvanishing only if we account for the rotational inertia of the attached object about its own centroid. The importance of (5) becomes clearer when we relax the constraint that the hoop rolls entirely without slipping.

Returning to (5), we make the following important observation: When hoop and supporting surface part company, the contact forces $n(t)$ and $f(t)$ vanish and (5) reduces to $\ddot{\theta}(t) = 0$. Thus, the hoop simply rotates about its center of mass at a constant angular speed equal to its angular speed at the instant the contact forces went to zero. At the same time, equations (3) and (4), which describe the translational motion of the center of mass, reduce to the equations for a body falling freely with a constant downward acceleration. This simultaneous free fall/free rotation continues until the height $y_c(t)$ of the center of the hoop decreases once again to R, at which time $n(t)$ acquires the positive value required to enforce constraint 2. In this way, we arrive at an alternative view of why the hoop hops: it is simply rotating about its center of mass, as that center of mass falls freely under the influence of gravity. Littlewood's hopping hoop is reminiscent of a good high jumper, who is skilled at rotating and "deforming" her body in such a way that all of its component parts clear the bar, even as her center of mass passes *under* it!

The motion of the system is completely determined by equations (3), (4), and (5), along with the relevant constraints and initial conditions. We have already noted that constraints 1, 2, and 3 are simultaneously satisfied only if the center of mass of the system moves along the trochoid:

$$x(t) = R(\theta(t) + \lambda \sin \theta(t)) \tag{6}$$

$$y(t) = R(1 + \lambda \cos \theta(t)) \tag{7}$$

By combining equations (3) through (7) one may obtain a single second-order differential equation for one of the variables ($\theta(t)$, say) from which the motion of the system may be determined. The result will be valid as long as the constraints embodied by (6) and (7) hold, i.e., as long as the hoop rolls without slipping along the supporting surface. Alternately, one may observe that the constraints imply that $x(t) = y(t)\dot{\theta}(t)$ and $\dot{y}(t) = (x(t) - x_c(t))\dot{\theta}(t)$, and use these relations to combine (3), (4), and (5) into a single equation, from which the forces of constraint $n(t)$ and $f(t)$ are absent and which, more importantly, may be written as a total derivative. In this way, one arrives at the equation of energy conservation. Physically, the absence of the forces of constraint from the equation of energy conservation reflects the fact that forces of constraint do no net work because they act perpendicular to the configuration space of the system. If now we assume, as Tokeida does, that the attached mass initially moves horizontally with speed v_0, energy conservation requires

$$\tfrac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2}I\dot{\theta}^2 + Mgy = M\left((1+\lambda)\left(\tfrac{1}{4}v_0^2 + gR\right) + \tfrac{1}{8}v_0^2\lambda\epsilon\right),$$

where the additional terms not seen in the corresponding equation in **[1]** arise from the rotational kinetic energy of the (massive) hoop and attached (extended) object. Adding the trochoid constraint, equations (6) and (7), we obtain

$$\dot{\theta}(t)^2 = \frac{g}{R}\frac{\left((1+\lambda+\lambda\epsilon/2)c + 2\lambda\sin^2(\theta(t)/2)\right)}{1+\lambda\cos(\theta(t)) + \lambda\epsilon/2} \tag{8}$$

where $c = v_0^2/(4gR)$. This result may be used in conjunction with (4) to compute $\ddot{y}(\theta(t))$, as in **[1]**. Tokieda observes that the hop must occur at the first value of $\theta(t)$ for which the parabolic trajectory of the center of mass in free fall departs from above from the trochoidal trajectory imposed by the rolling hoop, i.e., the hop occurs when $\ddot{y}(\theta(t)) = -g$. But, referring to (4), this is precisely when $n(t) = 0$, i.e., when the normal component of the force of contact between the hoop and the supporting surface vanishes, a condition that certainly *seems* to suggest that the hoop has just lost contact with the supporting surface and is about to become airborne. That is not the case, as we now prove.

Combining (4), (7), and (8), we obtain the following rather formidable expression for the normal force n as a function of $\xi = \cos\theta(t)$.

$$n(\xi) = \frac{g}{(\epsilon\lambda + 2\xi\lambda + 2)^2}p(\xi), \qquad \text{where}$$

$$\begin{aligned} p(\xi) = {} & (1 - c\lambda\xi)\lambda^2\epsilon^2 \\ & - \epsilon\lambda\left((c\xi^2 + 2c\xi + 2\xi + c - 3\xi^2 + 1)\lambda^2 + 4(c-1)\xi\lambda - 4\right) \\ & - 2\Big((c\xi^2 + \xi^2 + c - 2\xi^3 + 1)\lambda^3 \\ & \qquad + (c\xi^2 + 2c\xi + 2\xi + c - 5\xi^2 + 1)\lambda^2 + 2(c-2)\xi\lambda - 2\Big) \end{aligned} \tag{9}$$

We now specialize to the case of a massless hoop ($\lambda = 1$) and determine the angle θ_1 for which $n \to 0$ by finding the roots of the cubic polynomial $p(\xi)$ in (9). The result is most illuminating if we express it as a series expansion in the "inertia parameter" ϵ. The physically relevant root is

$$\cos\theta_1 = c - \frac{(1-c)}{(c+1)}\left(\frac{(c+3)}{4}\epsilon + \frac{1}{(c+1)^2}\epsilon^2\right) + O(\epsilon)^3. \tag{10}$$

If the attached object is a point mass ($\epsilon = 0$), we recover from (10) the result given in **[1]** for the critical angle θ_1 at which, it is claimed, the hoop loses contact with the supporting surface. If that is indeed the case, then the subsequent motion of the system must consist of a translation of the center of mass along the parabolic trajectory corresponding to free fall,

$$x(t) = x_1 + \dot{x}(t - t_1) \tag{11}$$

$$y(t) = y_1 + \dot{y}_1(t - t_1) - \tfrac{1}{2}g(t - t_1)^2, \tag{12}$$

combined with a rotation about the center of mass at constant angular speed $\dot{\theta}_1$:

$$\theta(t) = \theta_1 + \dot{\theta}_1(t - t_1). \tag{13}$$

Here, (x_1, y_1) and $(\dot{x}_1, \dot{y}_1)$ are the Cartesian coordinates of the position and velocity of the center of mass at the instant t_1 at which the normal force vanishes, and $\dot{\theta}_1$ is the corresponding angular speed of rotation. Series expansions of these quantities are obtained from (10), in conjunction with (6), (7), and (8). If one now substitutes (12) and (13) into (2), replacing y_1, $\dot{y}_1$, θ_1, and $\dot{\theta}_1$ by their expansions in ϵ, one obtains the following expression for the height $y_c[t]$ of the center of the hoop at times $t > t_1$:

$$\frac{y_c(\tau)}{R} = 1 - \frac{(1-c)((\epsilon+4)c^2 + 8c + 3\epsilon + 4)}{8(c+1)^2}\tau^2 + O(\epsilon)^2 + O(\tau)^3, \tag{14}$$

where $\tau = \sqrt{g/R}(t - t_1)$. Now, c must be strictly less than unity, or else the requirement that the hoop initially rolls without slipping cannot be satisfied; for $c \geq 1$ (i.e., for $v_0 \geq \sqrt{4gR}$), the hoop "glides" immediately without rolling, as Tokieda points out **[1]**. Since $c < 1$, the term of order τ^2 in (14) is negative. This means that an instant after the normal force between the hoop and the supporting surface goes to zero, the center of the hoop is moving not upward, but downward, a result that one could have obtained equally well by computing the acceleration $\ddot{y}_c(t_1)$! Thus, even though at t_1 the normal component of the contact force is instantaneously zero, it cannot remain zero for a nonvanishing duration. This is because at t_1 the center of the hoop is accelerating downward, and a positive normal force immediately results to enforce constraint 2 and prevent the height of the center of the hoop from falling below R. Equation (14) shows that a massless hoop cannot get off the ground if we insist that there be no slipping. In particular, this is true for the special case considered in **[1]** ($\epsilon = 0$: attached object is a point mass) as well as for Littlewood's original, formulation ($\epsilon = c = 0$: attached point mass with zero initial velocity) **[2]**. In the case of a massive hoop ($\lambda < 1$), the problem must be treated numerically, but the result is the same: as long as we require that the hoop roll entirely without slipping, it will never become airborne.

Even if the forces of constraint vanish instantaneously, the hoop and the supporting surface can part company only if $\dot{y} + R\dot{\theta}\sin\theta > 0$, i.e., only if the speed $R\dot{\theta}\sin\theta$ at which the center of the hoop rises (due to the rotation of the system about its center of mass) *exceeds* the speed $|\dot{y}|$ at which the center of mass falls under the influence of gravity. However, as long as we stipulate that the hoop rolls entirely without slipping, imposing by fiat the trochoid constraint, then it follows from (7) that the magnitude of the quantities $R\dot{\theta}\sin\theta$ and $\dot{y}$ are *equal.* Right on the money is Littlewood's **[2]** comment that, in practice, the hoop slips first (before hopping).

At what point does the hoop begin to slip? By combining equations (3), (6), and (8), one obtains the following expression for the force f responsible for maintaining the "no slipping" condition, constraint 3.

$$f(\xi) = -\frac{g\lambda\sqrt{1-\xi^2}}{(\lambda\epsilon + 2\lambda\xi + 2)^2}q(\xi), \qquad \text{where}$$

$$\begin{aligned} q(\xi) &= 2\epsilon\lambda^2 + (2\lambda - 3\epsilon\lambda + c(\epsilon\lambda + 2\lambda + 2) - 6)\xi\lambda + 2\lambda - \epsilon\lambda \\ &\quad + c(\epsilon^2\lambda^2 + 2\epsilon\lambda^2 + 3\epsilon\lambda + 2\lambda + 2) - 4\lambda^2\xi^2 - 2 \end{aligned} \tag{15}$$

We remind the reader that (9) and (15) represent, respectively, the vertical and horizontal components of the contact force necessary at any given angle $\theta = \arccos(\xi)$ to keep the center of mass moving on the trochoidal trajectory dictated by constraints 1, 2, and 3. It is natural to ask whether the required force is indeed physically available.

The normal force n is limited only by the resistance to tensile stress of the hoop and the supporting surface; in assuming that both the hoop and the surface are infinitely rigid, not subject to deformations of any kind, we allow n to be as large as is necessary to maintain constraint 2. In the real world, of course, no object is infinitely rigid. In the limit of small strains, however, the resulting deformations are elastic—energy-conserving—and the potential energy stored as elastic strain in the hoop and/or supporting surface just prior to the loss of contact provides an additional source of propulsion for the hop.

The frictional force f, on the other hand, has a maximum value, and if this is exceeded, slipping occurs. It is known experimentally that the maximum possible magnitude of the force of static friction existing at the point of contact of two surfaces is proportional to the magnitude of the normal force of contact exerted by one surface on the other:

$$|f| \leq |f_{\mathrm{MAX}}| = \mu_s|n| \tag{16}$$

The constant of proportionality, μ_s, is known as the *coefficient of static friction* and it depends on the nature of the surfaces that are in contact. For most materials, μ_s ranges from around 0.01 to 1.5 [**3**]. Once the point of contact between the hoop and the supporting surface begins to slip, the magnitude of the friction force is proportional to the magnitude of the normal force: $|f(t)| = \mu_k|n(t)|$. The relevant constant of proportionality, the coefficient of kinetic friction μ_k, is always less than the coefficient of static friction μ_s, for any pair of surfaces in contact. This is because the "cold-welding" that serves to bind the surfaces together in the static case cannot occur when there is relative motion between the two surfaces [**3**]. Once slipping begins, the friction force is directed so as to oppose the relative motion of the two surfaces involved. For the skidding hoop,

$$f(t) = \mu_k n(t)\,\mathrm{sgn}\left(R\dot{\theta}(t) - \dot{x}_c(t)\right), \tag{17}$$

where $\mathrm{sgn}(x)$ is the sign function and $n(t) \geq 0$. Specifically, $f(t)$ is positive (directed to the right) if the hoop is skidding forward, i.e., if the motion of the hoop relative to the point of contact is directed to the left, and $f(t)$ is negative (directed to the left) if the hoop is skidding backward, i.e., if the motion of the hoop relative to the point of contact is directed to the right.

The hoop begins to slip at the minimum angle for which equality holds in (16), i.e., for the minimum (real) root of $\{f(\cos\theta)\}^2 - \{\mu_s n(\cos\theta)\}^2 = 0$, where the forces of constraint n and f are given by (9) and (15), respectively. For example, in the case of a massless hoop ($\lambda = 1$) with attached point mass ($\epsilon = 0$), θ_{slip} is the lesser of $\arccos(c)$ and $2\arctan(\mu_s)$. The subsequent motion of the system is computed using (17) and numerically solving the coupled differential equations (3) and (5) subject to constraints 1 and 2. The former constraint (no deformations) translates simply into the twin conditions expressed by equations (1) and (2). Considerably more tedious to implement is the latter constraint, which must be enforced explicitly at each time step in the integration. One proceeds as follows: A provisional value of the normal force $n(t)$ is obtained from $n(t - \Delta t)$ by extrapolation. From $n(t)$, $x(t)$, $y(t)$, $\theta(t)$, $\dot{x}(t)$, $\dot{y}(t)$, and $\dot{\theta}(t)$ one computes a provisional value of $y_c(t + \Delta t)$. If $y_c(t + \Delta t) < R$, one adjusts $n(t)$ to give $y_c(t + \Delta t) = R$. The revised n is then used to compute the values of x, y, θ, $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ at time $t + \Delta t$. The procedure continues until $y_c(t)$ is strictly greater than R, at which point the hoop is airborne and the subsequent motion of the system is given by (11)–(13), where the subscript "1" now refers to values at the final step in the numerical procedure, i.e., at the instant contact between hoop and supporting surface is lost.

The implementation of the procedure just described is complicated by the fact that comparisons of approximate (floating point) quantities can be performed only to a certain tolerance. Thus, the stopping condition that y_c be strictly greater than R is, in practice, $y_c - R > \delta$; that is, we consider the hoop to be in contact with the supporting surface and, more importantly, we continue to impose constraints 1 and 2, until the point of contact clears the surface by at least δ. Since the parameter δ effectively *defines* when the hoop is considered to be airborne, the amount of time during which the hoop rolls while slipping prior to takeoff depends on δ, and so too does the height of the hop. Roughly speaking, this is because the longer the "rolling with slipping" phase prior to takeoff, the greater the takeoff speed that the hoop can build up—more precisely: the greater the amount by which $|R\dot{\theta} \sin \theta|$ can exceed $|\dot{y}|$. Here, $|\dot{y}|$ is the speed at which the center of mass falls under the influence of gravity, while $|R\dot{\theta} \sin \theta|$ is the speed at which the center of the hoop rises as a result of the rotation of the system about its centroid. Numerical simulations confirm that the height of the hop decreases with δ, so one might expect in the limit $\delta \to 0$ to observe no hop at all. However, the $\delta \to 0$ limit is not only in conflict with physical reality, it violates even the assumptions of this admittedly idealized problem! The point is simply this: Even if we were able to perform our numerical computations to infinite precision, δ could still not be made arbitrarily small. It is, after all, microscopic imperfections in the hoop and the supporting surface that gave rise to friction in the first place, and the scale of these imperfections provides a physical lower bound to δ. In the present case, we began with the assumption that hoop and surface are sufficiently "rough" that the former would initially roll without slipping over the latter. This precludes $\delta \to 0$.

The preceding discussion might lead one to doubt that the hop can actually be observed in practice. This is not the case: the hop is real, as the photograph in Figure 2 shows.

Figure 3 depicts schematically the results of a typical numerical simulation using parameter values for a system that one could potentially realize experimentally: the hoop is *not* massless, and the attached object is *not* an idealized point mass. Shown are the trajectory of the center of mass (black line) and the position of the center of mass at the onset of slipping and at the instant the hoop loses contact with the supporting surface (open circles). Dots indicate the positions of the attached object (large dots) and the center of the hoop (small dots) at twenty equally spaced times. Spokes (light gray lines) connecting simultaneous positions of the object and the center of the hoop have been added as an aid in visualization. The figure was generated using $\lambda = 0.95$, $\epsilon = 0.01$, and $c = 0.1$, the latter value corresponding to the attached object moving horizontally with an initial speed of 2 meters/sec. The coefficient of static and kinetic friction between the hoop and supporting surface were taken to have values 1.0 and 0.8, respectively. The simulation generating the figure used $\delta = R/100$; in simulations employing smaller values of δ, the hop is less evident, but it is always present.

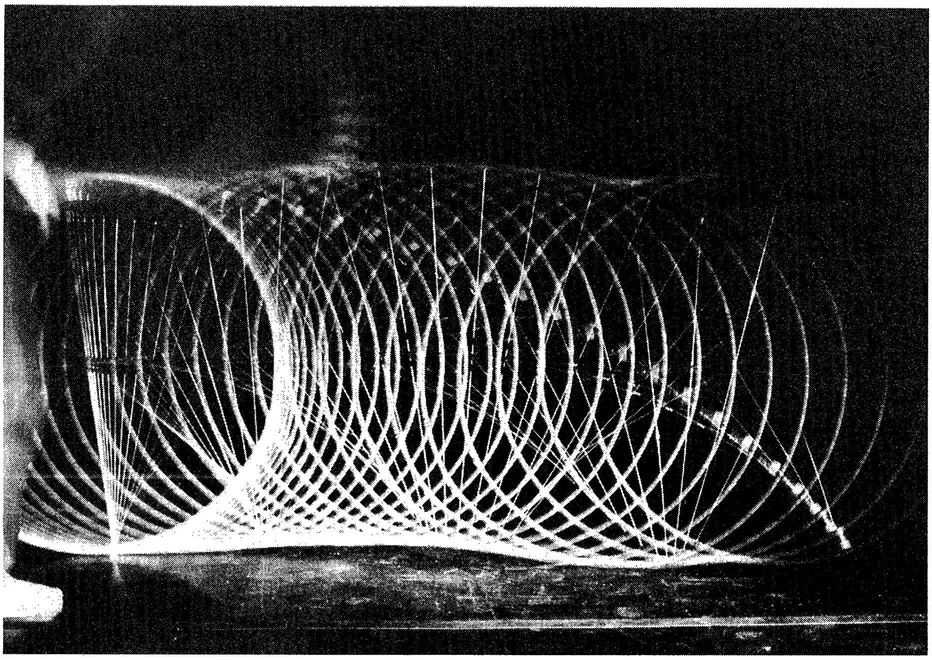

Figure 2. A stroboscopic photo by Dan Schwalbe and Stan Wagon shows a small hop of the hoop, which is a plastic hula hoop and four brass rods.

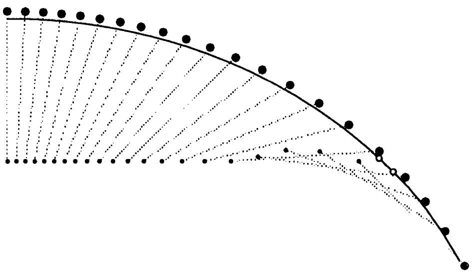

Figure 3. Simulated hop of a real hoop: results of a numerical simulation using $\lambda = 0.95$, $\epsilon = 0.01$, and $c = 0.1$ and coefficients of friction $\mu_s = 1.0$ and $\mu_k = 0.8$. The large dots indicate the positions of the attached object and the small dots indicate the corresponding positions of the center of the hoop. The trajectory of the center of mass of the system is indicated by the black line; the two open circles along the center of mass trajectory indicate the position of the center of mass at the onset of slipping and at the instant of loss of contact with the supporting surface.

We note in passing that if the hoop is massless ($\lambda = 0$) and the attached object is a point mass ($\epsilon = 0$), (8) can be integrated in closed form to give the following rather cute solution for $\theta(t)$:

$$\theta(t) = 2\arcsin\left[\sqrt{c}\,\sinh\left[\frac{t}{2}\sqrt{\frac{g}{R}}\right]\right] = 2\arcsin\left[\frac{v_0}{2\sqrt{gR}}\sinh\left[\frac{t}{2}\sqrt{\frac{g}{R}}\right]\right].$$

In his original essay [2], Littlewood gives his discussion of the hopping hoop a somewhat philosophical turn by posing a question to which we, in closing, now turn. Wonders Littlewood: Is the behavior of the hoop intuitive? We have demonstrated that there can be no hop without some slipping of the hoop. This conclusion may at first appear somewhat surprising. We hope, however, that the reader now agrees that the phenomenon of hopping can be quite intuitive, at least if it is explained in general terms. In this instance, as in countless others, the devil is in the details.

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