PROBLEMS AND SOLUTIONS

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with the collaboration of Paul T. Bateman, Mario Benedicty, Paul Bracken, Duane M. Broline, Ezra A. Brown, Richard T. Bumby, Glenn G. Chappell, Randall Dougherty, Roger B. Eggleton, Ira M. Gessel, Bart Goddard, Jerrold R. Griggs, Douglas A. Hensley, John R. Isbell, Robert Israel, Kiran S. Kedlaya, Murray S. Klamkin, Fred Kochman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Charles Vanden Eynden, and William E. Watkins.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before February 29, 2000; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

Problem **10743** [1999; 586] in the June–July 1999 issue was misstated. Here is the corrected version.

10743. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-La-Neuve, Belgium.* Let $R = \sum(-1)^i\binom{n}{i}$, where the sum is taken over all $i \in \{0, 1, \dots, n-1\}$ such that $i+1$ is a quadratic residue modulo p, and let $N = \sum(-1)^j\binom{n}{j}$, where the sum is taken over all $j \in \{0, 1, \dots, n-1\}$ such that $j+1$ is a quadratic nonresidue modulo p. Prove that exactly one of R and N is divisible by p.

PROBLEMS

10746. *Proposed by Stepan Tersian, University of Rousse, Rousse, Bulgaria.* Prove that

$$\int_0^\infty \left(e^{-y\sqrt{(s/x)^2+1}} - e^{-x\sqrt{(s/y)^2+1}}\right)\cos s\, ds = 0,$$

for all positive real numbers x and y.

10747. *Proposed by Athanasios Kalakos, Athens, Greece.* Find all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ that are twice differentiable on an open interval containing 0, have exactly one real root, satisfy $f(1)=1$, and satisfy $f'(f(t)) = 2f(t)$ for every $t\in\mathbb{R}$.

10748. *Proposed by Itshak Borosh, Douglas A. Hensley, and Joel Zinn, Texas A& M University, College Station, TX.* Let p and q be prime numbers, and let r be a positive integer such that $q\,|\,(p-1)$, $q \not| \, r$, and $p > r^{q-1}$. Show that for any integers $a_1, a_2, \dots, a_r$, if $\sum_{j=1}^r a_j^{(p-1)/q} \equiv 0 \bmod p$, then $\prod_{j=1}^r a_j \equiv 0 \bmod p$.

10749. *Proposed by Alain Grigis, Université Paris 13, Villetaneuse, France.* Let ABC be a triangle with a right angle at B and an angle of $\pi/6$ at A. Consider a billiard path in the triangle that begins at A, reflects successively off side BC at P, off side AC at Q, off side AB at R, off side AC at S, and then ends at B.

(**a**) Show that AP, QR, and SB are concurrent at a point X.

(**b**) Show that the angles formed at X measure $\pi/3$.

(**c**) Show that $AX = XP + PQ + QX = XR + RS + SX = 2XB$.