10750. *Proposed by Leonard Smiley, University of Alaska, Anchorage, AK.* For a positive integer m, express $\sum_{n=1}^{\infty}(n/\gcd(m,n))x^n$ as a rational function of x.

10751. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* Let n be a positive integer, and let S_n be the set of all strings $a_1a_2\cdots a_n$ of positive integers satisfying $a_1 = 1$ and $a_{i+1} - a_i \in \{1, -1, -3, -5, \ldots\}$. For example, $S_5 = \{12345, 12343, 12341, 12323, 12321, 12123, 12121\}$. Find $|S_n|$.

10752. *Proposed by Gh. Costovici, Technical University "Gh. Asachi", Iasi, Romania.* For $n \in \mathbb{N}$, let a_n and b_n be complex numbers, with each $b_n \neq 0$. Let $s_n = a_1 + a_2 + \cdots + a_n$, and let $t_n = (1 - b_1/b_{n+1})\,a_1 + (1 - b_2/b_{n+1})\,a_2 + \cdots + (1 - b_n/b_{n+1})\,a_n$.
(**a**) Prove that if $\lim_{n\to\infty} b_{n+1}/b_n = 1$ and $\sum_{n=1}^{\infty}|s_n - t_n|^q$ converges for some $q \in (0, 1]$, then $\sum_{n=1}^{\infty} a_n$ converges.
(**b**) Prove that if $\sum_{n=1}^{\infty}|b_{n+1}/b_n - 1|^r$ and $\sum_{n=1}^{\infty}|s_n - t_n|^{r/(r-1)}$ converge for some $r \in (1, \infty)$, then $\sum_{n=1}^{\infty} a_n$ converges.

SOLUTIONS

A Zeta Function over a Recurrent Sequence

10486 [1995, 841]. *Proposed by Joseph H. Silverman, Brown University, Providence, RI.* Let $a, b > 0$ and $\alpha > 1$ be real numbers, and define $Z(s) = \sum_{n\in\mathbb{Z}}\left(a\alpha^n + b\alpha^{-n}\right)^{-s}$ for complex numbers s with positive real part.
(**a**) Prove that $Z(s)$ has a meromorphic continuation to all of $\mathbb{C}$.
(**b**) Find the poles of $Z(s)$.
(**c**) Find the residues of $Z(s)$ at its poles.

Solution I by David Bradley, University of Maine, Orono, ME. Let σ be the real part of s. Write

$$Z(s) = (a+b)^{-s} + \sum_{n=1}^{\infty}\left(a\alpha^n + b\alpha^{-n}\right)^{-s} + \sum_{n=1}^{\infty}\left(b\alpha^n + a\alpha^{-n}\right)^{-s}. \tag{1}$$

Without loss of generality, assume that $0 < a \le b$. We first consider the case $|\alpha| > \sqrt{b/a}$. We then have the two binomial expansions

$$\left(a\alpha^n + b\alpha^{-n}\right)^{-s} = \frac{a^{-s}\alpha^{-ns}}{\left(1 + ba^{-1}\alpha^{-2n}\right)^s} = a^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{b^k}{a^k}\alpha^{-2nk} + E_{m,n}(s)\right) \tag{2}$$

and

$$\left(b\alpha^n + a\alpha^{-n}\right)^{-s} = \frac{b^{-s}\alpha^{-ns}}{\left(1 + ab^{-1}\alpha^{-2n}\right)^s} = b^{-s}\alpha^{-ns}\left(\sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^k}{b^k}\alpha^{-2nk} + F_{m,n}(s)\right), \tag{3}$$

where m is a fixed positive integer and $E_{m,n}(s) = O(\alpha^{-2mn})$ and $F_{m,n}(s) = O(\alpha^{-2mn})$. Since $|\alpha| > \sqrt{b/a}$, it follows from (1)–(3) that

$$\begin{aligned} Z(s) &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\left(\frac{b^k}{a^{s+k}} + \frac{a^k}{b^{s+k}}\right)\sum_{n=1}^{\infty}\alpha^{-n(s+2k)} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right) \\ &= (a+b)^{-s} + \sum_{k=0}^{m-1}\binom{-s}{k}\frac{a^{-s-k}b^k + b^{-s-k}a^k}{\alpha^{s+2k} - 1} + O\left(\sum_{n=1}^{\infty}\alpha^{-n(\sigma+2m)}\right). \end{aligned} \tag{4}$$

Since $E_{m,n}(s)$ and $F_{m,n}(s)$ are analytic for $\sigma > -2m$, it follows by analytic continuation that (4) is valid for $\sigma > -2m$. Since m is an arbitrary positive integer, we conclude that $Z(s)$ has a meromorphic continuation to the entire complex plane.

We now calculate the poles and residues from (4). For integer n, let $s_n = 2\pi i n/\log\alpha$. From (4), we see that all singularities of $Z(s)$ are simple poles, and these occur at points of the form $s_n - 2k$, where n is an integer and k is a nonnegative integer. The residue of $Z(s)$ at the pole $s_n - 2k$ is $\binom{2k-s_n}{k} a^k b^k (a^{-s_n} + b^{-s_n})/\log\alpha$.

If $(b/a)^{2n}$ is an odd power of α, then $s_n - 2k$ is an ordinary point, not a pole. Finally, although our derivation of (4) is valid only for complex numbers α with $|\alpha| > \sqrt{b/a}$, this restriction can be eased to $|\alpha| > 1$ by analytic continuation in α, provided that m remains fixed.

Solution II by Donald A. Darling, Newport Beach, CA. With $\beta = \log\alpha$, $\gamma = \frac{1}{2}\log(a/b)$, and $\delta = \sqrt{ab}$, the function $Z(s)$ takes the form

$$Z(s) = \frac{1}{(2\delta)^s} \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^s(n\beta + \gamma)}.$$

Since the real part of s is positive, the function $\cosh^{-s}(n\beta + \gamma)$ satisfies the hypotheses of the Poisson summation formula $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\cos(2\pi n x)dx$. Thus,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{1}{\cosh^s(n\beta + \gamma)} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos(2\pi n x)}{\cosh^s(\beta x + \gamma)} dx = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos(2\pi n(y - \gamma/\beta))}{\cosh^s(\beta y)} dy \\
&= \sum_{n=-\infty}^{\infty} \cos\left(\frac{2\pi n\gamma}{\beta}\right) \int_{-\infty}^{\infty} \frac{\cos(2\pi n y)}{\cosh^s(\beta y)} dy + \sum_{n=-\infty}^{\infty} \sin\left(\frac{2\pi n\gamma}{\beta}\right) \int_{-\infty}^{\infty} \frac{\sin(2\pi n y)}{\cosh^s(\beta y)} dy \\
&= \sum_{n=-\infty}^{\infty} \cos\left(\frac{2\pi n\gamma}{\beta}\right) \int_{-\infty}^{\infty} \frac{\cos(2\pi n y)}{\cosh^s(\beta y)} dy. \qquad (5)
\end{aligned}$$

Formula 3.985(1) (p. 540) of I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 1980 states

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi n y)}{\cosh^s(\beta y)} dy = \frac{2^{s-1}}{\beta\Gamma(s)} \Gamma\left(\frac{s}{2} + \frac{\pi i n}{\beta}\right) \Gamma\left(\frac{s}{2} - \frac{\pi i n}{\beta}\right).$$

With (5), this yields

$$\begin{aligned}
Z(s) &= \frac{1}{2\beta\Gamma(s)\delta^s} \sum_{n=-\infty}^{\infty} \cos\left(\frac{2\pi n\gamma}{\beta}\right) \Gamma\left(\frac{s}{2} + \frac{\pi i n}{\beta}\right) \Gamma\left(\frac{s}{2} - \frac{\pi i n}{\beta}\right) \\
&= \frac{\Gamma^2(s/2)}{2\beta\Gamma(s)\delta^s} + \frac{1}{\beta\Gamma(s)\delta^s} \sum_{n=1}^{\infty} \cos\left(\frac{2\pi n\gamma}{\beta}\right) \Gamma\left(\frac{s}{2} + \frac{\pi i n}{\beta}\right) \Gamma\left(\frac{s}{2} - \frac{\pi i n}{\beta}\right).
\end{aligned} \qquad (6)$$

By Stirling's formula for the gamma function in a vertical strip, the series in the last term of (6) converges uniformly in any vertical strip bounded away from the poles of the summands. Thus, $Z(s)$ has an analytic continuation to the entire complex s-plane, and the poles and residues may be calculated as in the first solution.

Editorial comment. When $|\alpha| > \sqrt{b/a}$, we can let $m \to \infty$ in (4) to obtain

$$Z(s) = (a + b)^{-s} + \sum_{k=0}^{\infty} \binom{-s}{k} \frac{a^{-s-k}b^k + b^{-s-k}a^k}{\alpha^{s+2k} - 1},$$

which is valid for all complex s.

When $a = b = 1/2$ and $\alpha = e^{\pi z}$ with the real part of z positive, (6) yields $Z(s) = \sum_{n=-\infty}^{\infty} \cosh^{-s}(n\pi z)$. For positive integer s, such sums arise in the theory of elliptic functions. For even s (C. B. Ling, On summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 5 (1974) 551–561) and for all positive integral s (I. J. Zucker, The summation of series of hyperbolic functions, *SIAM J. Math. Anal.* 10 (1979) 192–206), $Z(s)$ can be evaluated in terms of elliptic functions. In particular, for $s = 1$, $0 < k < 1$, and $z = K(\sqrt{1-k^2})/K(k)$, we have $\sum_{n=-\infty}^{\infty} \cosh^{-1}(n\pi z) = (\sum_{n=-\infty}^{\infty} e^{-\pi n^2 z})^2 = (2/\pi)K(k)$, where $K(k)$ is the complete elliptic integral of the first kind (see B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, 1991, p. 102 and p. 138).

Solved also by D. Cantor, R. J. Chapman (U. K.), R. Holzsager, and the proposer.

A Matrix of Inequalities

10599 [1997, 566]. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be nonnegative numbers and let (a_{ij}) be an $m \times n$ matrix of nonnegative numbers with at least one nonzero entry in each row. Suppose that the inequality $\sum_{h=1}^{m} a_{hj}x_h \le \sum_{k=1}^{n} a_{ik}y_k$ holds whenever $a_{ij} > 0$. Show that $\sum_{i=1}^{m} x_i \le \sum_{j=1}^{n} y_j$.

Solution by Frank Jelen and Eberhard Triesch, Der Rheinisch-Westfälischen Technischen Hochschule, Aachen, Germany. Let A be the specified matrix, with columns $c_1, \dots, c_n$. Let $x = (x_1, \dots, x_m)^T$ and $y = (y_1, \dots, y_n)^T$, and let $\mathbf{1}_k$ denote the column vector of length k with entries equal to 1.

Define $b = (b_1, \dots, b_m)^T$ by $b_i = \max\{c_j^T x\colon a_{ij} > 0\}$; this is well-defined since each row contains a positive entry. Consider the linear programs

$$\text{minimize} \quad \mathbf{1}_n^T z \qquad \text{subject to} \quad Az \ge b \text{ and } z \ge 0 \tag{1}$$

and

$$\text{maximize} \quad b^T w \qquad \text{subject to} \quad A^T w \le \mathbf{1}_n \text{ and } w \ge 0. \tag{2}$$

These linear programs are duals of each other, and (1) has the feasible solution $z = y$. It thus suffices to show that there exists a feasible solution u of (2) with $b^T u \ge \mathbf{1}_m^T x$, since the Duality Theorem then yields $\mathbf{1}_n^T y \ge b^T u \ge \mathbf{1}_m^T x$.

Consider the nonnegative vector $u = (u_1, \dots, u_m)^T$ defined by $u_i = x_i/b_i$ if $b_i > 0$ and $u_i = 0$ otherwise. Clearly $b^T u = \mathbf{1}_m^T x$.

For $1 \le j \le n$, define $I_j = \{i : a_{ij} > 0 \text{ and } x_i > 0\}$. For $i \in I_j$, we have $b_i \ge c_j^T x > 0$. Feasibility of u now follows from

$$c_j^T u = \sum_{i=1}^{m} a_{ij}u_i = \sum_{i \in I_j} a_{ij}\frac{x_i}{b_i} \le \frac{1}{c_j^T x}\sum_{i \in I_j} a_{ij}x_i = 1.$$

Solved also by the proposer.

A Complex Determinant

10601 [1997, 566]. *Proposed by Wen-Xiu Ma, Universität-GH Paderborn, Paderborn, Germany.* Let $n > 1$ be an integer and let $a_1, a_2, \dots, a_n$ be complex numbers. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{2n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{2n-1} \\ \vdots & & & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{2n-1} \\ 0 & 1 & 2a_1 & \cdots & (2n-1)a_1^{2n-2} \\ 0 & 1 & 2a_2 & \cdots & (2n-1)a_2^{2n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & 1 & 2a_n & \cdots & (2n-1)a_n^{2n-2} \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{1 \le i < j \le n} (a_i - a_j)^4.$$