



## On a Convolution of Eulerian Numbers: 10609

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### On a Convolution of Eulerian Numbers

**10609** [1997, 664]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let  $a(l, m, n) = \sum_{k=0}^l \binom{n}{k} (l+m-k)^{n-k} (k-l)^k$ . Prove that  $\sum_{l=1}^n a(l, m, n) = ((m+n+1)/2)a(n, m, n) - (m+1)/2 m^n$ .

*Solution by David Callan, University of Wisconsin, Madison, WI.* We compare coefficients of  $m^j$  to prove the desired identity. First we express  $a(l, m, n)$  using Eulerian numbers. The classical Eulerian number  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$  is the number of permutations of  $[n] = \{1, \dots, n\}$  consisting of  $k$  ascending runs (with descents at  $k-1$  locations). These are counted by placing  $n$  numbered objects into  $k$  numbered boxes to avoid properties  $P_1, \dots, P_k$ , where  $P_i$  is the property that box  $i$  is empty or has only objects greater than those in the preceding box. A careful application of the inclusion-exclusion principle yields the formula  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k-j)^n$ . Note that  $\left\langle \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\rangle = 1$ .

The definition of  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$  yields  $\sum_{k=0}^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = n!$ . We also need  $\sum_{k=0}^n k \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = \frac{1}{2}(n+1)!$ . To prove this combinatorially, we alter each permutation of  $[n]$  with  $k$  runs by placing  $n+1$  at the end of one run. This can be done in  $k$  ways and yields a permutation of  $[n+1]$ . A permutation of  $[n+1]$  arises in this way if  $n+1$  is at the end, not if  $n+1$  is at the start, and otherwise if and only if the element preceding  $n+1$  is greater than the element following it. Thus we obtain half the permutations of  $[n+1]$ .

We claim that

$$a(l, m, n) = \sum_{j=0}^n \sum_{i=0}^l \binom{n}{j} \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle m^j, \quad (*)$$

which we prove by comparing coefficients of  $m^j$ . By applying the binomial theorem to  $(l-k+m)^{n-k}$ , we extract the desired coefficient as  $\sum_{k=0}^l (-1)^k \binom{n}{k} \binom{n-k}{j} (l-k)^{n-j}$ . Rearranging binomial coefficients converts this to  $\binom{n}{j} \sum_{k=0}^l (-1)^k \binom{n-j}{k} (l-k)^{n-j}$ . After canceling the  $\binom{n}{j}$ , it remains only to rewrite the sum as  $\sum_{i=0}^l \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle$ . We use our formula for Eulerian numbers, let  $k = i - h$ , apply the elementary identity  $\sum_{h=0}^r (-1)^h \binom{n+1}{h} = (-1)^r \binom{n}{r}$ , and finally interchange  $k$  with  $l-k$  to obtain

$$\begin{aligned} \sum_{i=0}^l \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle &= \sum_{i=0}^l \sum_{h=0}^i (-1)^h \binom{n-j+1}{h} (i-h)^{n-j} = \sum_{k=0}^l k^{n-j} \sum_{h=0}^{l-k} (-1)^h \binom{n-j+1}{h} \\ &= \sum_{k=0}^l (-1)^{l-k} \binom{n-j}{l-k} k^{n-j} = \sum_{k=0}^l (-1)^k \binom{n-j}{k} (l-k)^{n-j}. \end{aligned}$$

Using (\*), we compute coefficients of  $m^j$  in the desired identity. For  $j = n+1$ , the contributions cancel. For  $j = n$ , the coefficient is  $n$ . For  $j < n$ , the coefficient of  $m^j$  in  $\sum_{l=1}^n a(l, m, n)$  divided by  $\binom{n}{j}$  is

$$\begin{aligned} \sum_{l=1}^n \sum_{i=1}^l \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle &= \sum_{i=1}^n (n+1-i) \sum_{i=1}^n \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle = (n+1) \sum_{i=1}^n \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle - \sum_{i=1}^n i \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle \\ &= (n+1)(n-j)! - \frac{1}{2}(n-j+1)! = (n-j)! \frac{n+j+1}{2}. \end{aligned}$$

For  $j < n$ , the coefficient of  $m^j$  in  $((m+n+1)/2)a(n, m, n) - (m+1)/2 m^n$  is

$$\begin{aligned} \frac{n+1}{2} \binom{n}{j} \sum_{i=1}^n \left\langle \begin{smallmatrix} n-j \\ i \end{smallmatrix} \right\rangle + \frac{1}{2} \binom{n}{j-1} \sum_{i=1}^n \left\langle \begin{smallmatrix} n-j+1 \\ i \end{smallmatrix} \right\rangle \\ = \frac{n+1}{2} \binom{n}{j} (n-j)! + \frac{1}{2} \binom{n}{j-1} (n-j+1)! = \binom{n}{j} (n-j)! \frac{n+j+1}{2}. \end{aligned}$$

*Editorial comment.* From (\*) we infer that  $a(l, 1, n)$  is the total number of arrangements with at most  $l$  ascending runs that can be formed from subsets of  $[n]$ .

Solved also by R. J. Chapman (U. K.), Q. Darwish (Oman), H.-J. Seiffert (Germany), and the proposer.

### A Variation on Additive Bases

**10610** [1997, 664]. *Proposed by Richard Hall, University of Portsmouth, Portsmouth, England.* Given a positive integer  $m$ , let  $C(m)$  be the greatest positive integer  $k$  such that, for some set  $S$  of  $m$  integers, every integer from 1 to  $k$  belongs to  $S$  or is a sum of two not necessarily distinct elements of  $S$ . For example,  $C(3) = 8$  with  $S = \{1, 3, 4\}$ .

(a) Show that, for all  $\epsilon > 0$ ,  $1/4 < C(m)/m^2 < 1/2 + \epsilon$  for all sufficiently large  $m$ .

(b)\* Improve the asymptotic bounds in part (a).

*Solution to (a) by the National Security Agency Problems Group, Fort Meade, MD.* Let  $[n]_l$  denote the first  $n$  positive multiples of  $l$ . When  $m$  is even, with  $m = 2t$ , let  $S = [t-1]_1 \cup [t+1]_t$ . Since  $S$  has size  $m$  and represents all positive integers up to  $(t+1)t + t$ , we have  $C(m) \geq t^2 + 2t$ . Thus  $C(m)/m^2 \geq (t^2 + 2t)/(2t)^2 > 1/4$ .

When  $m$  is odd, with  $m = 2t + 1$ , let  $S = [t]_1 + [t+1]_t(t+1)$ . Since  $S$  has size  $m$  and represents all positive integers up to  $(t+1)^2 + t + 1$ , we have  $C(m) \geq (t+1)(t+2)$ . Thus  $C(m)/m^2 \geq (t+1)(t+2)/(2t+1)^2 > 1/4$ .

A set of size  $m$  represents at most  $2m + \binom{m}{2}$  integers. Hence  $C(m)/m^2 \leq 1/2 + 3/(2m) < 1/2 + \epsilon$  for  $m > 3/(2\epsilon)$ .

*Solution to (b) by the GCHQ Problems Group, Cheltenham, UK.* We show that  $9/32 < C(m)/m^2 < 4/9 + \epsilon$  for all sufficiently large  $m$ .

For the lower bound, we construct a set that represents many integers by spreading the summands apart more quickly than in (a). Write  $m$  as  $16i + j$ , where  $-7 \leq j \leq 8$ , and let  $A = [1, 3i]$ ,  $B = [2, 7i + j]_{3i}$ ,  $C = (7i + j)_{3i} + [1, 3i]_{(3i+1)}$ , and  $D = (7i + j)_{6i} + 6i + [0, 3i]$ , where  $[x, y] = \{n \in \mathbb{Z} : x \leq n \leq y\}$ . Let  $S = A \cup B \cup C \cup D$ .

From  $A$  and  $A + A$  we get  $[1, 6i]$ , from  $A + B$  we get  $[6i + 1, (7i + j)_{3i} + 3i]$ , and from  $C$  and  $A + C$  we get  $[(7i + j)_{3i} + 3i + 1, 30i^2 + 3i(j + 2)]$ .

For  $r \in [1, 4i + j]$  and  $s \in [1, 3i]$ , we have  $(3i + r - s + 1)_{3i} \in B$  and  $(7i + j)_{3i} + s(3i + 1) \in C$ , and the sum of these integers is  $(10i + r + j + 1)_{3i} + s$ . Thus  $B + C$  contains  $[(10i + 2 + j)_{3i} + 1, (14i + 2j + 1)_{3i} + 3i]$ , which equals  $[30i^2 + 3i(j + 2) + 1, (7i + j)_{6i} + 6i]$ . Furthermore,  $D \cup (A + D) \cup (B + D) = [(7i + j)_{6i} + 6i, (7i + j)_{9i} + 9i]$ , and  $C + D = [(7i + j)_{9i} + 9i + 1, (7i + j)_{9i} + 3i(3i + 1) + 9i] = [(7i + j)_{9i} + 9i + 1, 72i^2 + i(12 + 9j)]$ .

Since  $|A \cup B \cup C \cup D| = m$ , for large enough  $i$  we have

$$\frac{C(m)}{m^2} \geq \frac{72i^2 + i(12 + 9j)}{256i^2 + 32ij + j^2} = \frac{9}{32} \frac{8i^2 + i(j + 4/3)}{8i^2 + ij + j^2/32} > \frac{9}{32}.$$

To prove that  $C(m)/m^2 < 4/9 + \epsilon$ , we show that some of the  $m + \binom{m}{2}$  pairs must be "wasted". This happens in two ways. First, the sum may be too big, as happens for any pair of numbers that both exceed  $C(m)/2$ . Second, note that  $r - s = t - u$  if and only if  $r + u = t + s$ . Thus we obtain a wasted pair for each instance of identical differences.

Consider a set  $S$  that represents everything from 1 to  $\mu m^2$ , for some  $\mu > 1/4$ . We may assume that  $S \subseteq [1, \mu m^2]$ . Let  $am = |S \cap [1, \mu m^2/2]|$ . All pairs from the  $(1 - a)m$  numbers above  $\mu m^2/2$  are wasted. The smaller pairs have differences between 1 and  $\mu m^2/2 - 1$ , yielding wastage when  $am + \binom{am}{2} > \mu m^2/2 - 1$ .

Let  $a \geq b$  mean that  $a > b - \epsilon$  for large enough  $m$ . Letting  $wm^2$  be the number of wasted pairs, we have  $w \geq \max(0, (a^2 - \mu)/2) + (1 - a)^2/2$ . Letting  $f(a)$  denote this lower bound, we have  $f'(a) = -(1 - a)$  for  $a^2 < \mu$  and  $f'(a) = 2a - 1$  for  $a^2 > \mu$ . The first quantity is negative and the second positive, since  $\mu > 1/4$ . Thus  $w$  is minimized at