On a Convolution of Eulerian Numbers

10609 [1997, 664]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $a(l,m,n) = \sum_{k=0}^{l} \binom{n}{k}(l+m-k)^{n-k}(k-l)^k$. Prove that $\sum_{l=1}^{n} a(l,m,n) = \big((m+n+1)/2\big)a(n,m,n) - \big((m+1)/2\big)m^n$.

Solution by David Callan, University of Wisconsin, Madison, WI. We compare coefficients of m^j to prove the desired identity. First we express $a(l,m,n)$ using Eulerian numbers. The classical Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $[n] = \{1,\dots,n\}$ consisting of k ascending runs (with descents at $k-1$ locations). These are counted by placing n numbered objects into k numbered boxes to avoid properties $P_1,\dots,P_k$, where P_i is the property that box i is empty or has only objects greater than those in the preceding box. A careful application of the inclusion-exclusion principle yields the formula $\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n$. Note that $\left\langle {0 \atop 0} \right\rangle = 1$.

The definition of $\left\langle {n \atop k} \right\rangle$ yields $\sum_{k=0}^{n}\left\langle {n \atop k} \right\rangle = n!$. We also need $\sum_{k=0}^{n} k\left\langle {n \atop k} \right\rangle = \frac{1}{2}(n+1)!$. To prove this combinatorially, we alter each permutation of $[n]$ with k runs by placing $n+1$ at the end of one run. This can be done in k ways and yields a permutation of $[n+1]$. A permutation of $[n+1]$ arises in this way if $n+1$ is at the end, not if $n+1$ is at the start, and otherwise if and only if the element preceding $n+1$ is greater than the element following it. Thus we obtain half the permutations of $[n+1]$.

We claim that

$$a(l,m,n) = \sum_{j=0}^{n}\sum_{i=0}^{l}\binom{n}{j}\left\langle {n-j \atop i} \right\rangle m^j, \qquad (*)$$

which we prove by comparing coefficients of m^j. By applying the binomial theorem to $(l-k+m)^{n-k}$, we extract the desired coefficient as $\sum_{k=0}^{l}(-1)^k\binom{n}{k}\binom{n-k}{j}(l-k)^{n-j}$. Rearranging binomial coefficients converts this to $\binom{n}{j}\sum_{k=0}^{l}(-1)^k\binom{n-j}{k}(l-k)^{n-j}$. After canceling the $\binom{n}{j}$, it remains only to rewrite the sum as $\sum_{i=0}^{l}\left\langle {n-j \atop i} \right\rangle$. We use our formula for Eulerian numbers, let $k = i-h$, apply the elementary identity $\sum_{h=0}^{r}(-1)^h\binom{n+1}{h} = (-1)^r\binom{n}{r}$, and finally interchange k with $l-k$ to obtain

$$\sum_{i=0}^{l}\left\langle {n-j \atop i} \right\rangle = \sum_{i=0}^{l}\sum_{h=0}^{i}(-1)^h\binom{n-j+1}{h}(i-h)^{n-j} = \sum_{k=0}^{l}k^{n-j}\sum_{h=0}^{l-k}(-1)^h\binom{n-j+1}{h}$$
$$= \sum_{k=0}^{l}(-1)^{l-k}\binom{n-j}{l-k}k^{n-j} = \sum_{k=0}^{l}(-1)^k\binom{n-j}{k}(l-k)^{n-j}.$$

Using $(*)$, we compute coefficients of m^j in the desired identity. For $j = n+1$, the contributions cancel. For $j = n$, the coefficient is n. For $j < n$, the coefficient of m^j in $\sum_{l=1}^{n} a(l,m,n)$ divided by $\binom{n}{j}$ is

$$\sum_{l=1}^{n}\sum_{i=1}^{l}\left\langle {n-j \atop i} \right\rangle = \sum_{i=1}^{n}(n+1-i)\sum_{i=1}^{n}\left\langle {n-j \atop i} \right\rangle = (n+1)\sum_{i=1}^{n}\left\langle {n-j \atop i} \right\rangle - \sum_{i=1}^{n} i\left\langle {n-j \atop i} \right\rangle$$
$$= (n+1)(n-j)! - \frac{1}{2}(n-j+1)! = (n-j)!\frac{n+j+1}{2}.$$

For $j < n$, the coefficient of m^j in $\big((m+n+1)/2\big)a(n,m,n) - \big((m+1)/2\big)m^n$ is

$$\frac{n+1}{2}\binom{n}{j}\sum_{i=1}^{n}\left\langle {n-j \atop i} \right\rangle + \frac{1}{2}\binom{n}{j-1}\sum_{i=1}^{n}\left\langle {n-j+1 \atop i} \right\rangle$$
$$= \frac{n+1}{2}\binom{n}{j}(n-j)! + \frac{1}{2}\binom{n}{j-1}(n-j+1)! = \binom{n}{j}(n-j)!\frac{n+j+1}{2}.$$

Editorial comment. From $(*)$ we infer that $a(l, 1, n)$ is the total number of arrangements with at most l ascending runs that can be formed from subsets of $[n]$.

Solved also by R. J. Chapman (U. K.), Q. Darwish (Oman), H.-J. Seiffert (Germany), and the proposer.

A Variation on Additive Bases

10610 [1997, 664]. *Proposed by Richard Hall, University of Portsmouth, Portsmouth, England.* Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S. For example, $C(3) = 8$ with $S = \{1, 3, 4\}$.
(**a**) Show that, for all $\epsilon > 0$, $1/4 < C(m)/m^2 < 1/2 + \epsilon$ for all sufficiently large m.
(**b**)* Improve the asymptotic bounds in part (**a**).

Solution to (**a**) *by the National Security Agency Problems Group, Fort Meade, MD.* Let $[n]l$ denote the first n positive multiples of l. When m is even, with $m = 2t$, let $S = [t-1]1 \cup [t+1]t$. Since S has size m and represents all positive integers up to $(t+1)t + t$, we have $C(m) \geq t^2 + 2t$. Thus $C(m)/m^2 \geq (t^2 + 2t)/(2t)^2 > 1/4$.

When m is odd, with $m = 2t + 1$, let $S = [t]1 + [t+1](t+1)$. Since S has size m and represents all positive integers up to $(t+1)^2 + t + 1$, we have $C(m) \geq (t+1)(t+2)$. Thus $C(m)/m^2 \geq (t+1)(t+2)/(2t+1)^2 > 1/4$.

A set of size m represents at most $2m + \binom{m}{2}$ integers. Hence $C(m)/m^2 \leq 1/2 + 3/(2m) < 1/2 + \epsilon$ for $m > 3/(2\epsilon)$.

Solution to (**b**) *by the GCHQ Problems Group, Cheltenham, UK.* We show that $9/32 < C(m)/m^2 < 4/9 + \epsilon$ for all sufficiently large m.

For the lower bound, we construct a set that represents many integers by spreading the summands apart more quickly than in (**a**). Write m as $16i + j$, where $-7 \leq j \leq 8$, and let $A = [1, 3i]$, $B = [2, 7i + j]3i$, $C = (7i + j)3i + [1, 3i](3i + 1)$, and $D = (7i + j)6i + 6i + [0, 3i]$, where $[x, y] = \{n \in \mathbb{Z}\colon x \leq n \leq y\}$. Let $S = A \cup B \cup C \cup D$.

From A and $A + A$ we get $[1, 6i]$, from $A + B$ we get $[6i + 1, (7i + j)3i + 3i]$, and from C and $A + C$ we get $[(7i + j)3i + 3i + 1, 30i^2 + 3i(j + 2)]$.

For $r \in [1, 4i + j]$ and $s \in [1, 3i]$, we have $(3i + r - s + 1)3i \in B$ and $(7i + j)3i + s(3i + 1) \in C$, and the sum of these integers is $(10i + r + j + 1)3i + s$. Thus $B + C$ contains $[(10i + 2 + j)3i + 1, (14i + 2j + 1)3i + 3i]$, which equals $[30i^2 + 3i(j + 2) + 1, (7i + j)6i + 6i]$. Furthermore, $D \cup (A + D) \cup (B + D) = [(7i + j)6i + 6i, (7i + j)9i + 9i]$, and $C + D = [(7i + j)9i + 9i + 1, (7i + j)9i + 3i(3i + 1) + 9i] = [(7i + j)9i + 9i + 1, 72i^2 + i(12 + 9j)]$.

Since $|A \cup B \cup C \cup D| = m$, for large enough i we have

$$\frac{C(m)}{m^2} \geq \frac{72i^2 + i(12 + 9j)}{256i^2 + 32ij + j^2} = \frac{9}{32}\frac{8i^2 + i(j + 4/3)}{8i^2 + ij + j^2/32} > \frac{9}{32}.$$

To prove that $C(m)/m^2 < 4/9 + \epsilon$, we show that some of the $m + \binom{m}{2}$ pairs must be "wasted". This happens in two ways. First, the sum may be too big, as happens for any pair of numbers that both exceed $C(m)/2$. Second, note that $r - s = t - u$ if and only if $r + u = t + s$. Thus we obtain a wasted pair for each instance of identical differences.

Consider a set S that represents everything from 1 to μm^2, for some $\mu > 1/4$. We may assume that $S \subseteq [1, \mu m^2]$. Let $am = \left|S \cap [1, \mu m^2/2]\right|$. All pairs from the $(1 - a)m$ numbers above $\mu m^2/2$ are wasted. The smaller pairs have differences between 1 and $\mu m^2/2 - 1$, yielding wastage when $am + \binom{am}{2} > \mu m^2/2 - 1$.

Let $a \succeq b$ mean that $a > b - \epsilon$ for large enough m. Letting wm^2 be the number of wasted pairs, we have $w \succeq \max(0, (a^2 - \mu)/2) + (1 - a)^2/2$. Letting $f(a)$ denote this lower bound, we have $f'(a) = -(1 - a)$ for $a^2 < \mu$ and $f'(a) = 2a - 1$ for $a^2 > \mu$. The first quantity is negative and the second positive, since $\mu > 1/4$. Thus w is minimized at