Uniform Calculus and the Law of Bounded Change

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1. INTRODUCTION. In a recent exchange about the role of the mean value theorem in the theory of the calculus, T. Tucker notes that "the origin of the Mean Value Theorem in the structure of the real numbers" is much too difficult for a standard course [**6**]. He shows how the increasing function theorem (a function with positive derivative is increasing) serves very nicely in place of the mean value theorem, and sketches a proof of it from the nested interval property of the real number system.

In support of the mean value theorem, H. Swann recalls its derivation from the extreme value theorem (a continuous function on a closed interval has a maximum value) via Rolle's theorem and remarks that "such a sequence of arguments reveals the charm and power of mathematics, for we prove that a questionable complicated result *must* be true if we assume other simpler results that are less questionable" [**5**].

We agree with Swann about the charm and power of mathematics and with Tucker about the ability of the increasing function theorem to play a role traditionally accorded the mean value theorem. In fact, we give several examples that support Tucker's claim. But Tucker and Swann work with *pointwise* continuity and differentiability, weak notions that make proving statements like the increasing function theorem more difficult. On closed finite intervals, uniform continuity and differentiability are as easy to verify, and using them as starting points permits a natural development of the calculus in which such difficulties do not arise.

Our treatment of continuity and differentiation is from our forthcoming book, *A New Course of Analysis*, where it is expressed in terms of a theory of real numbers based on interval order and arithmetic. We offer no theory of real numbers in this article but we use repeatedly the fact that each real number can be approximated by rationals to arbitrary accuracy.

2. UNIFORM CALCULUS. Continuity. Uniform continuity of a one variable function f is a condition on its variation, $f(y) - f(x)$. The condition, written $\phi(x, y) \to 0$ as $y - x \to 0$ for any two-variable function ϕ, is that for each $\epsilon > 0$, there is a $\delta > 0$ such that $|\phi(x, y)| \leq \epsilon$ if $|y - x| \leq \delta$. When $\phi(x, y) = U(x, y) - u(x)$, we also write: $U(x, y) \to u(x)$ as $y \to x$.

Example. The relationship

$$y^n - x^n = \left(\sum_{i=1}^{n} y^{n-i} x^{i-1} \right) (y - x)$$

shows that $|y^n - x^n| \leq nC^{n-1} |y - x|$ on any interval of the form $[-C, C]$ and, hence, that x^n is uniformly continuous on each finite interval. A proof of pointwise continuity could hardly be simpler.

Example. Using $p^n + q^n \le (p+q)^n$ with $p = x^{1/n}$ and $q = (y-x)^{1/n}$, we have

$$0 \le y^{1/n} - x^{1/n} \le (y-x)^{1/n} \le \epsilon \quad \text{if } 0 \le x \le y \quad \text{and } y - x \le \epsilon^n.$$

It follows that for each positive integer n, $x^{1/n}$ is uniformly continuous on $[0, \infty)$.

Proposition 2.1. *A composition of uniformly continuous functions is uniformly continuous.*

Proposition 2.2. *A uniformly continuous function f on a finite interval I is bounded.*

Proof: For $\epsilon > 0$, let $\delta > 0$ be given by uniform continuity. Because I is finite, we can find finitely many points such that every $p \in I$ is within δ of at least one of them. Hence, f is bounded by ϵ plus the maximum of its values at these finitely many points. ■

Differentiability. Uniform differentiability of a function f also is a condition on its variation: it factors as $f(y) - f(x) = F(x, y)(y - x)$, where $F(x, y) \to F(x, x)$ as $y \to x$. If f is uniformly differentiable, its derivative is the function $f'(x) = F(x, x)$. Thus, $F(x, y) = (f(y) - f(x))/(y - x)$, for y different from x, and $F(x, x) = f'(x)$.

Because the difference quotient converges to the derivative as $y \to x$, the derivative is unique on any domain S for which each x in S is approximable to arbitrary accuracy by points y in S different from x.

Example. For all positive integers n, using the factorization of $y^n - x^n$ and the arithmetic of convergence (see Lemma 4.1), it follows that on each finite interval, x^n is uniformly differentiable with derivative nx^{n-1}.

Example. Because $y^2 - x^2 = (y + x)(y - x)$ and $y + x \to 2x$ on $\mathbb{R}$ as $y \to x$, x^2 is uniformly differentiable on $\mathbb{R}$ with derivative $2x$.

Proposition 2.3. *If f is uniformly differentiable, then f' is uniformly continuous.*

Proof: Because F is symmetric, if x and y are close enough, both $f'(x)$ and $f'(y)$ are within ϵ of $F(x, y) = F(y, x)$ and hence within 2ϵ of each other. ■

Corollary 2.4. *On finite intervals, f' is bounded.*

Proof: Propositions 2.2 and 2.3. ■

Proposition 2.5. *If f' is bounded, then f is uniformly continuous.*

Proof: When f' is bounded, so is $F(x, y)$, say by C, for $|y - x|$ sufficiently small. Hence, $|f(y) - f(x)| \le C|y - x| \to 0$ as $y - x \to 0$. ■

Theorem 2.6. (*Fundamental Theorem of the Calculus*) *If g is uniformly continuous on $[a, b]$, then $G(x) \equiv \int_a^x g(t)\,dt$ is uniformly differentiable on $[a, b]$ with $G' = g$.*

Proof: $G(y) - G(x)$ equals the integral of g from x to y, which equals $y - x$ times a limit of averages of values of g at points in $[x, y]$. (To see this, approximate the integral by Riemann sums with equal spacing.) Also, for each $\epsilon > 0$, if $|y - x|$

is small enough, every value of g at a point in $[x, y]$ is within ϵ of $g(x)$. But then, also, any limit of averages of values of g at points in $[x, y]$ is within ϵ of $g(x)$, so we are done.

3. THE ARITHMETIC OF UNIFORM CONTINUITY. The arithmetic of uniform continuity is very simple. If both f and g are uniformly continuous, so is $f + g$. If also f and g are bounded, then fg is uniformly continuous. Finally, if $1/f$ is defined and bounded, it too is uniformly continuous.

These statements can be verified by first relating the variations of the sum, product and reciprocal to those f and g. Simple algebra shows that $\text{var}(f+g) = \text{var}(f) + \text{var}(g)$, $\text{var}(fg) = g(y)\text{var}(f) + f(x)\text{var}(g)$ and $\text{var}(1/f) = -\text{var}(f)/(f(x)f(y))$.

Because each expression is a sum of expressions of the form $B(x, y)\phi(x, y)$, where $B(x, y)$ is bounded and $\phi(x, y) \to 0$ as $y - x \to 0$, it suffices to verify that each such sum again converges to 0 as $y - x \to 0$. We omit the simple proof of this.

4. THE ARITHMETIC OF UNIFORM DIFFERENTIABILITY. If f and g are uniformly differentiable, derivatives for their arithmetic combinations are given by the following rules.

Sums. $f + g$ is uniformly differentiable with $(f+g)' = f' + g'$.

Products. If f, g, and their derivatives are bounded, e.g., if their domain is a finite interval, then fg is uniformly differentiable with $(fg)' = f'g + fg'$.

Reciprocals. If $1/f$ is defined and bounded, and f' also is bounded, then $1/f$ is uniformly differentiable with $(1/f)' = -f'/f^2$.

To prove these assertions, we begin by substituting $F(x, y)(y - x)$ and $G(x, y)(y - x)$ for $\text{var}(f)$ and $\text{var}(g)$ in our expressions for $\text{var}(f+g)$, $\text{var}(fg)$, and $\text{var}(1/f)$. For the sum, we get $F(x, y) + G(x, y)$, for the product, $g(y)F(x, y) + f(x)G(x, y)$, and for the reciprocal, $-F(x, y)/(f(x)f(y))$, each multiplied by $y - x$.

For $y = x$, these expressions become $f'(x) + g'(x)$, $g(x)f'(x) + f(x)g'(x)$, and $-f'(x)/f^2(x)$.

The case of the sum is clear. Both $F(x, y) - f'(x)$ and $G(x, y) - g'(x)$ converge to 0 as $y - x \to 0$, hence so does the sum. For the product and reciprocal, multiplications are involved. The following lemma gives us what we need to deal with them.

Lemma 4.1. *Suppose that u and v are bounded. If $U(x, y) \to u(x)$ and $V(x, y) \to v(x)$ as $y \to x$, then for $y - x$ sufficiently small, U and V are bounded and $U(x, y)V(x, y) \to u(x)v(x)$ as $y \to x$.*

Proof: Write $UV - uv = u(V - v) + (U - u)V$ and note that, because u and V are bounded for $y - x$ small enough, each summand converges to 0 as $y - x \to 0$. ∎

The next lemma is used to prove Proposition 6.1 about the differentiability of an inverse function.

Lemma 4.2. *Suppose that $1/u$ is defined and bounded. If $U(x, y) \to u(x)$, then for $y - x$ sufficiently small, $1/U$ is defined and bounded, and $1/U(x, y) \to 1/u(x)$ as $y \to x$.*

Proof: We prove only the second part. Write $1/u - 1/U = (U - u)/uU$ and note that $1/uU$ is bounded for $y - x$ sufficiently small. ■

For the product rule, we reason as follows. By assumption, f is bounded and $G(x, y) \to g'(x)$ as $y \to x$. Thus, $f(x)G(x, y) \to f(x)g'(x)$ as $y \to x$. Because limits add, it suffices to prove that $g(y)F(x, y) \to g(x)f'(x)$ as $y \to x$. But, also by assumption, $F(x, y) \to f'(x)$ as $y \to x$, and f' and g are bounded. Hence, if $g(y) \to g(x)$ as $y \to x$, we can apply Lemma 4.1. It therefore suffices to note that g is uniformly continuous because g' is bounded.

Similarly, for the reciprocal rule, f is uniformly continuous because f' is bounded, and because $1/f$ is bounded, it too is uniformly continuous. Hence, $1/f(y) \to 1/f(x)$ as $y \to x$. Multiplying by $1/f(x)$, we see that $1/f(x)f(y) \to 1/f^2(x)$ as $y \to x$. Because $-F(x, y) \to -f'(x)$ as $y \to x$, and the limit functions $1/f^2$ and $-f'$ are bounded, the product converges to the product by Lemma 4.1.

5. THE CHAIN RULE

Proposition 5.1. *If f and g are uniformly differentiable, and if f' and g' are bounded, then $f(g)$ is uniformly differentiable with derivative $f'(g)g'$.*

Proof: Because $f(g(y)) - f(g(x)) = F(g(x), g(y))(g(y) - g(x))$, which in turn equals $F(g(x), g(y))G(x, y)(y - x)$, the candidate for the derivative of $f(g)$ is indeed $f'(g)g'$. Because g' is bounded, g is uniformly continuous. Hence, $F(g(x), g(y)) \to f'(g(x))$ as $y \to x$. Because $G(x, y) \to g'(x)$ as $y \to x$ and $f'(g)$ is bounded, an application of Lemma 4.1 gives us the desired result. ■

6. DIFFERENTIABILITY OF THE INVERSE

Proposition 6.1. *If h is a uniformly continuous inverse for f, and if $1/f'$ is defined and bounded, then h is uniformly differentiable with $h' = 1/f'(h)$.*

Proof: Because h is an inverse for f, we can factor the variation of the identity function as

$$y - x = f(h(y)) - f(h(x)) = F(h(x), h(y))(h(y) - h(x)).$$

This shows that $1/F(h(x), h(y))$ is equal to the difference quotient for h when $|y - x| > 0$. Because h is uniformly continuous, $F(h(x), h(y)) \to f'(h(x))$ as $y - x \to 0$. Therefore, because $1/f'(h(x))$ is defined and bounded, we can apply Lemma 4.2 to conclude that $1/F(h(x), h(y)) \to 1/f'(h(x))$ as $y \to x$. ■

7. THE LAW OF BOUNDED CHANGE

Theorem 7.1. *If f is uniformly differentiable and $A \le f' \le B$ on $[a, b]$, then $A(b - a) \le f(b) - f(a) \le B(b - a)$.*

This is the law of bounded change. It says that bounds for the derivative are bounds for the difference quotient. Notice that the increasing function theorem is just the law of bounded change for $A = 0$ (and we don't care about B) and the law of bounded change is the increasing function theorem applied to the functions $Bx - f(x)$ and $f(x) - Ax$.

Proof: It suffices to prove that for all $\epsilon > 0$, the conclusion holds with A and B replaced by $A - \epsilon$ and $B + \epsilon$. The justification for this is the general truth that if $p < q + \epsilon$ for all $\epsilon > 0$, then $p \le q$. That this holds for reals follows by rational approximation from the fact that it holds for rationals.

Since $F(u, v) \to f'(u)$ as $v \to u$, for each $\epsilon > 0$ there is a $\delta > 0$ such that $f'(u) - \epsilon < F(u, v) < f'(u) + \epsilon$ for $0 \le v - u < \delta$. But $A \le f'(u) \le B$, so $f(v) - f(u) = F(u, v)(v - u)$ lies between $(A - \epsilon)(v - u)$ and $(B + \epsilon)(v - u)$.

Hence, if we express $f(b) - f(a)$ as a telescoping sum of n differences $f(u_i) - f(u_{i-1})$, where $u_0 = a$ and each $u_i - u_{i-1} = (b - a)/n < \delta$, we have that $(A - \epsilon)(b - a) \le f(b) - f(a) \le (B + \epsilon)(b - a)$. ■

We now draw several useful and easy consequences of the law of bounded change.

Corollary 7.2. *f is constant on any interval on which $f' = 0$.*

Proof: This is just the law of bounded change with A and B equal to 0. ■

Is there any simpler or essentially different way to prove this deceptively obvious-looking fact?

Corollary 7.3. *$f(x) - f(a) = \int_a^x f'(t)\,dt$.*

Proof: By the fundamental theorem of the calculus, the two sides of the equation have the same derivative. Hence, by Corollary 7.2, they differ by a constant. But they agree at $x = a$, so they agree everywhere. ■

Alternatively, we can observe that in the proof of the law of bounded change, we in effect approximate $f(x) - f(a)$ to arbitrary accuracy by Riemann sums for the integral of f' from a to x. Because these sums also approximate the integral, the two must be equal.

Corollary 7.4. *If $f' \ge A > 0$ on $[a, b]$ and $f(h(u)) = u$ for all u in $[f(a), f(b)]$, then h is uniformly continuous.*

Proof: By the law of bounded change, if $h(u) < h(v)$, then $A(h(v) - h(u)) \le f(h(v)) - f(h(u)) = v - u$. So $0 < h(v) - h(u) \le (v - u)/A \to 0$ as $v - u \to 0$. ■

By the inverse function theorem, whenever $f' \ge A > 0$ on $[a, b]$, there is a function h as in the statement of Corollary 7.4.

Corollary 7.5. *If $A \le f' \le B$ on $[a, b]$, then*

$$\left| \frac{f(y) - f(x)}{y - x} - f'(u) \right| \le B - A \quad \text{for all } x < y \text{ and all } u \text{ in } [a, b].$$

Proof: We apply the law of bounded change on $[x, y]$. Because the values of f' are in $[A, B]$, so is the difference quotient $(f(y) - f(x))/(y - x)$, which therefore cannot differ from any value of f' by more than $B - A$. ■

Corollary 7.6. *If f is uniformly differentiable on all sufficiently small subintervals of an interval I and if f' is uniformly continuous on I, then f is uniformly differentiable on I.*

Proof: For $\epsilon > 0$, the values of f' lie between $f'(x) - \epsilon$ and $f'(x) + \epsilon$ on each sufficiently small $[x, y]$ in I. Therefore, if $f(y) - f(x) = F(x, y)(y - x)$, Corollary 7.5 shows that $|F(x, y) - f'(x)| \leq 2\epsilon$ for $|y - x|$ sufficiently small. ■

The next consequence of the law of bounded change is needed for L'Hôpital's Rule. In it, A and B are constants, and f and g are uniformly differentiable on $[a, b]$.

Corollary 7.7. (*Generalized Law of Bounded Change*) *If $Ag' \leq f' \leq Bg'$ on $[a, b]$, then $A[g(b) - g(a)] \leq f(b) - f(a) \leq B[g(b) - g(a)]$.*

Proof: Apply the increasing function theorem to $Bg - f$ and $f - Ag$, and rearrange the resulting inequalities. ■

8. APPLICATION: L'HÔPITAL'S RULE. We now present a few examples in support of Tucker's contention that the increasing function theorem serves nicely to prove major theorems of the calculus that traditionally are derived from the mean value theorem [**6**]. We begin with L'Hôpital's Rule; see also [**2**]. There are two cases. In both, we assume that f and g are defined on a semi-infinite interval $[c, \infty)$ and are uniformly differentiable on each finite subinterval. We assume also that g and g' are positive.

Proposition 8.1. *If $f(x)$ and $g(x) \to 0$ and $f'(x)/g'(x) \to L$ as $x \to \infty$, then also $f(x)/g(x) \to L$ as $x \to \infty$.*

Proof: For $\epsilon > 0$, $L - \epsilon \leq f'/g' \leq L + \epsilon$ on $[p, \infty)$ if p is large enough. In that case, if $p \leq x \leq y$, the generalized law of bounded change ensures that

$$(L - \epsilon)(g(x) - g(y)) \leq f(x) - f(y) \leq (L + \epsilon)(g(x) - g(y)).$$

Because weak inequalities are preserved in the limit, if we let $y \to \infty$ and divide by $g(x) > 0$, we obtain $L - \epsilon \leq f(x)/g(x) \leq L + \epsilon$ for all $x \geq p$. ■

In the second case of L'Hôpital's Rule, it is common to assume also that $f(x) \to \infty$, but there is no need to do so.

Proposition 8.2. *If $g(x) \to \infty$ and $f'(x)/g'(x) \to L$ as $x \to \infty$, then also $f(x)/g(x) \to L$ as $x \to \infty$.*

Here too, the generalized law of bounded change is used only once. We note that if $f'(x)/g'(x)$ lies between $L - \epsilon/2$ and $L + \epsilon/2$ for $x \geq p$, then so does $(f(x) - f(p))/(g(x) - g(p))$. But to complete the argument, one has to be more artful than in the first case.

9. APPLICATION: DIFFERENTIATION UNDER THE INTEGRAL SIGN

Definition 9.1. A two-variable function f is *uniformly continuous* if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x', y') - f(x, y)| \leq \epsilon$ whenever both $|x' - x|$ and $|y' - y|$ are smaller than δ.

Remark 9.2. If the second coordinates in Definition 9.1 are equal, then the condition that guarantees that $|f(x', y) - f(x, y)| \leq \epsilon$ involves only the first coordinates: $|x' - x| \leq \delta$. That is, for each $\epsilon > 0$, one $\delta > 0$ works for all horizontal lines y = constant. This simple observation is a key to our proof of Theorem 9.3.

Theorem 9.3. (*Differentiation Under the Integral Sign*) *Let f be defined on $Q = [a, b] \times [c, d]$ and uniformly continuous on $\{a\} \times [c, d]$. If f is uniformly differentiable on each $[a, b] \times \{y\}$ and its partial derivative f_x is uniformly continuous on Q, then f is uniformly continuous on Q and the integral of f_x over $[c, d]$ is a derivative for the integral of f over $[c, d]$.*

Proof: We assume the uniform continuity of f on Q. It can be be proved fairly easily using Corollary 7.3 but we prefer to focus here on the second part of the argument, which employs a less familiar application of the law of bounded change. Integrating $f(y, t) - f(x, t) = F(x, y, t)(y - x)$ over $[c, d]$, we see that to complete the proof, it suffices to demonstrate that the integral of $F(x, y, t) - f_x(x, t)$ over $[c, d]$ converges to 0 as $y - x \to 0$. To this end, it suffices to show that $|F(x, y, t) - f_x(x, t)|$ can be made less than any $\epsilon > 0$ by making $|y - x|$ less than some $\delta > 0$, independent of t.

It follows from Corollary 7.5 that $|F(x, y, t) - f_x(x, t)|$ is bounded for each t by any bound for $|f_x(v, t) - f_x(u, t)|$, for all u and v in $[x, y]$. By Remark 9.2, for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - y| \leq \delta$, then, for all t in $[c, d]$, $|f_x(v, t) - f_x(u, t)| \leq \epsilon$ for all u and v in $[x, y]$. This is precisely what we need. ■

Proposition 9.3 also follows easily from reversal of order of integration and Corollary 7.3. Because reversal of order of integration is a simple consequence of the existence of the double integral of a uniformly continuous function, this provides a proof of Proposition 9.3 that uses the law of bounded change in a more familiar way.

10. HIGHER DIMENSIONS. In higher dimensions, there is no obvious counterpart to the increasing function theorem, and the mean value theorem is false even for a mapping from an interval to $\mathbb{R}^2$. Yet the law of bounded change generalizes almost without alteration if we regard convex sets as higher dimensional counterparts to intervals and read the proof as showing that if f is defined on an interval $[a, b]$, then any closed interval that contains $\{f'(u)(b - a) : u \in [a, b]\}$ also contains $f(b) - f(a)$.

Definition 10.1. A map f from a subset U of one normed linear space X to another Y is *uniformly differentiable* if there is a map Df from U to the set of bounded linear transformations from X to Y such that for each $\epsilon > 0$, $\|f(q) - f(p) - Df(p)(q - p)\|_Y \leq \epsilon \|q - p\|_X$ if $\|q - p\|_X$ is sufficiently small.

Proposition 10.2. *Using the notation in Definition* 10.1, *if U is convex then, for each p and q in U, $f(q) - f(p)$ belongs to every convex subset of Y that contains $Df(u)(q - p)$ for all u in U. Hence, if each $Df(u): X \to Y$ is bounded by K on the unit sphere of X, then $\|f(q) - f(p)\|_Y \leq K\|q - p\|_X$.*

Proof: If p and q are in U, so is the line segment joining them. Hence, using a telescoping sum as in the proof of Theorem 7.1 and approximating each summand by the value of Df at a point on the segment, applied to $(q-p)/n$, we can approximate $f(q)-f(p)$ to arbitrary accuracy by an average of finitely many values of Df at points along the segment, applied to $q-p$. ■

11. AFTERWORD. We believe that this development, which is in the constructivist manner of Errett Bishop and L. E. J. Brouwer **[4]**, produces proofs that are shorter and more transparent than those encountered in classical treatments. The idea of working with uniform rather than pointwise notions is a hallmark of the constructivist tradition.

For the one-dimensional case, our definition of differentiable function is a uniform version of a definition of Carathéodory. See **[3]** and the references therein. For a definition of this kind in higher dimensions, see **[1]**.

REFERENCES

1. Acosta, E. and Delgado, C., Fréchet vs. Carathéodory, *Amer. Math. Monthly* **101** (1994) 332–338.
2. Boas, R.P., L'Hôpital's Rule without Mean Values, *Amer. Math. Monthly* **76** (1969) 1051–1053.
3. Kuhn, Stephen, The Derivative á la Carathéodory, *Amer. Math. Monthly* **98** (1991) 40–44.
4. Stolzenberg, Gabriel, review of *Foundations of Constructive Analysis* by Errett Bishop, *Bull. Amer. Math. Soc.* **76** (1970) 301–323.
5. Swann, Howard, Commentary on Rethinking Rigor in Calculus: The Role of the Mean Value Theorem, *Amer. Math. Monthly* **104** (1997) 241–245.
6. Tucker, Thomas, Rethinking Rigor in Calculus: The Role of the Mean Value Theorem, *Amer. Math. Monthly* **104** (1997) 231–240.

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