Conditional Convergence of Infinite Products

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In this article we revisit the classical subject of infinite products. For standard definitions and theorems on this subject see [**1**] or almost any textbook on complex analysis. We will restate parts of this material required to set the stage for our results, as follows.

The infinite product $P = \prod^{\infty}(1 + a_n)$ of complex numbers is said to *converge* if there is an integer N such that $1 + a_n \neq 0$ for $n \geq N$ and $\lim_{n\to\infty}\prod_{m=N}^{n}(1 + a_m)$ is finite and nonzero. This occurs if and only if the series $\sum_{m=N}^{\infty}\log(1 + a_m)$ converges.

We say that P *converges absolutely* if $\prod^{\infty}(1 + |a_n|)$ converges. If P converges absolutely then P converges, but the converse is false. The following theorem [**1**, p. 223] settles the question of absolute convergence of infinite products.

Theorem 1. *The infinite product* $\prod^{\infty}(1 + a_n)$ *converges absolutely if and only if* $\sum^{\infty}|a_n| < \infty$.

If P converges but $\prod^{\infty}(1 + |a_n|)$ does not, then we say that P *converges conditionally.* Conditional convergence of $\sum^{\infty}a_n$ does not imply conditional convergence of P. The following theorem [**1**, p. 225] seems to be the only general result along these lines, at least in the textbook literature.

Theorem 2. *If* $\sum^{\infty}|a_n|^2 < \infty$ *then* $\sum^{\infty}a_n$ *and* $\prod^{\infty}(1 + a_n)$ *converge or diverge together.*

Here we offer some other results concerning convergence of infinite products. Because of Theorem 1, these results are of interest only in the case where $\sum^{\infty}|a_n| = \infty$.

Theorem 3. *If there is a sequence* $\{r_n\}$ *such that*

$$\lim_{n\to\infty} r_n = 1 \tag{1}$$

and

$$\sum^{\infty}|r_n(1 + a_n) - r_{n+1}| < \infty, \tag{2}$$

then $\prod^{\infty}(1 + a_n)$ *converges.*

Proof: Let $g_n = r_n(1 + a_n) - r_{n+1}$. Then

$$\sum^{\infty}|g_n| < \infty \tag{3}$$

from (2), so $\lim_{n\to\infty}g_n = 0$ and therefore $\lim_{n\to\infty}a_n = 0$ by (1). Choose N so that r_n, $1 + a_n$ and $1 + g_n/r_{n+1}$ are nonzero if $n \geq N$. Now define $p_{N-1} = 1$ and

$$p_n = \prod_{m=N}^{n}(1 + a_m), \quad n \geq N.$$

If $n \geq N$ then $1 + a_n = p_n/p_{n-1}$, so $g_n = (r_n p_n/p_{n-1}) - r_{n+1}$, and therefore $p_n = r_{n+1} p_{n-1}(1 + g_n/r_{n+1})/r_n$, which implies that

$$p_n = \frac{r_{n+1}}{r_N} \prod_{m=N}^{n} (1 + g_m/r_{m+1}). \tag{4}$$

Since (1) and (3) imply that $\Sigma^\infty |g_m/r_{m+1}| < \infty$, Theorem 1 implies that the infinite product

$$Q = \prod_{m=N}^{\infty} (1 + g_m/r_{m+1})$$

converges; moreover $Q \neq 0$ because $1 + g_m/r_{m+1} \neq 0$ if $m \geq N$. Now (1) and (4) imply that $\lim_{n\to\infty} p_n = Q/r_N$ is finite and nonzero. ∎

To apply this theorem we must exhibit a sequence $\{r_n\}$ that will enable us to obtain results even if $\Sigma^\infty |a_n| = \infty$. The following theorem provides a way to do this.

Theorem 4. *Suppose that for some positive integer q the sequences*

$$a_n^{(k)} = \sum_{m=n}^{\infty} a_m a_m^{(k-1)}, \quad k = 1, \ldots, q \ \left(\text{with } a_m^{(0)} = 1\right),$$

are all defined, and

$$\sum^{\infty} \left| a_n a_n^{(q)} \right| < \infty. \tag{5}$$

Then $\prod^\infty (1 + a_n)$ converges.

Proof: Define

$$r_n^{(k)} = 1 + \sum_{j=1}^{k} (-1)^j a_n^{(j)}, \quad 1 \leq k \leq q.$$

We show by finite induction on k that

$$r_n^{(k)}(1 + a_n) - r_{n+1}^{(k)} = (-1)^k a_n a_n^{(k)} \tag{6}$$

for $1 \leq k \leq q$. Since $\lim_{n\to\infty} r_n^{(q)} = 1$ we can then set $k = q$ and conclude from (5) and Theorem 3 with $r_n = r_n^{(q)}$ that $\prod^\infty (1 + a_n)$ converges.

Since $r_n^{(1)} = 1 - a_n^{(1)}$ the left side of (6) with $k = 1$ is

$$\left(1 - a_n^{(1)}\right)(1 + a_n) - \left(1 - a_{n+1}^{(1)}\right) = a_n - a_n^{(1)} - a_n a_n^{(1)} + a_{n+1}^{(1)} = -a_n a_n^{(1)},$$

since $a_{n+1}^{(1)} + a_n = a_n^{(1)}$. This proves (6) for $k = 1$.

Now suppose that (6) holds if $1 \leq k < q - 1$. Since $r_n^{(k)} = r_n^{(k+1)} + (-1)^k a_n^{(k+1)}$, (6) implies that

$$\left(r_n^{(k+1)} + (-1)^k a_n^{(k+1)}\right)(1 + a_n) - r_{n+1}^{(k+1)} - (-1)^k a_{n+1}^{(k+1)} = (-1)^k a_n a_n^{(k)}.$$

Therefore

$$\begin{aligned} r_n^{(k+1)}(1 + a_n) - r_{n+1}^{(k+1)} &= (-1)^k \left(a_n a_n^{(k)} - a_n^{(k+1)} - a_n a_n^{(k+1)} + a_{n+1}^{(k+1)}\right) \\ &= (-1)^{(k+1)} a_n a_n^{(k+1)}, \end{aligned}$$

since $a_{n+1}^{(k+1)} + a_n a_n^{(k)} = a_n^{(k+1)}$. This completes the induction. ∎

We now prepare for a specific application of Theorem 4. Henceforth Δ is the forward difference operator; thus, if $\{g_m\}$ is a sequence, then $\Delta g_m = g_{m+1} - g_m$, while if G is a function of the continuous variable x then $\Delta G(x) = G(x+1) - G(x)$. Higher order forward differences are defined inductively; thus, if $\nu \geq 2$ is an integer, then

$$\Delta^{\nu} g_m = \Delta^{\nu-1} g_{m+1} - \Delta^{\nu-1} g_m = \sum_{r=0}^{\nu} (-1)^{r-\nu} \binom{\nu}{r} g_{m+r}.$$

A similar definition yields $\Delta^{\nu} G(x)$.

Lemma 1. *Suppose that t is a real number, not an integral multiple of 2π, and $\{g_m\}_{m=0}^{\infty}$ is a sequence such that $\lim_{m\to\infty} g_m = 0$ and*

$$\sum^{\infty} |\Delta^{\nu} g_m| < \infty \tag{7}$$

for some positive integer ν. Then $\Sigma^{\infty} g_m e^{imt}$ converges and

$$\sum_{m=0}^{\infty} g_m e^{imt} = (1-e^{it})^{-\nu} \left[\sum_{s=0}^{\nu-1} A_s g_s + e^{i\nu t} \sum_{m=0}^{\infty} (\Delta^{\nu} g_m) e^{imt} \right], \tag{8}$$

where

$$A_s = \sum_{m=s}^{\nu-1} (-1)^{m-s} \binom{\nu}{m-s} e^{imt}, \quad 0 \leq s \leq \nu - 1. \tag{9}$$

Proof: Suppose that $M > 2\nu$ and let

$$S_M = (1-e^{it})^{\nu} \sum_{m=0}^{M} g_m e^{imt}. \tag{10}$$

Since

$$(1-e^{it})^{\nu} e^{imt} = \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} e^{i(m+r)t},$$

we have

$$S_M = \sum_{m=0}^{M} g_m \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} e^{i(m+r)t} = \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} \sum_{m=0}^{M} g_m e^{i(m+r)t}$$
$$= \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} \sum_{m=r}^{M+r} g_{m-r} e^{imt}.$$

Reversing the order of summation in the last sum yields

$$S_M = \sum_{m=0}^{\nu-1} \left(\sum_{r=0}^{m} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt} + \sum_{m=\nu}^{M} \left(\sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt}$$
$$+ \sum_{m=M+1}^{M+\nu} \left(\sum_{r=m-M}^{\nu} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt}.$$

Since $\lim_{m\to\infty} g_m = 0$ the last sum on the right converges to 0 as $M \to \infty$. The second sum on the right is

$$\sum_{m=\nu}^{M} (\Delta^{\nu} g_{m-\nu}) e^{imt} = e^{i\nu t} \sum_{m=0}^{M-\nu} (\Delta^{\nu} g_m) e^{imt},$$

which converges as $M \to \infty$ because of (7). Therefore

$$\lim_{M\to\infty} S_M = S \equiv \sum_{m=0}^{\nu-1} \left(\sum_{r=0}^{m} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt} + e^{i\nu t} \sum_{m=0}^{\infty} (\Delta^{\nu} g_m) e^{imt},$$

which can also be written as

$$S = \sum_{s=0}^{\nu-1} A_s g_s + e^{i\nu t} \sum_{m=0}^{\infty} (\Delta^\nu g_m) e^{imt},$$

with A_s as in (9). This and (10) imply (8). ■

Henceforth we write $G(x) = O(x^{-\alpha})$ to indicate that $x^\alpha G(x)$ remains bounded as $x \to \infty$.

Definition 1. Let $\mathscr{F}_\alpha$ be the set of infinitely differentiable functions F on $[1, \infty)$ such that

$$F^{(\nu)}(x) = O(x^{-\alpha-\nu}), \quad \nu = 0, 1, \dots \tag{11}$$

For example, let $F(x) = u^\gamma(x)$, where u is a rational function with positive values on $[1, \infty)$ and a zero of order $p > 0$ at ∞; then F satisfies (11) with $\alpha = p\gamma$. To see this, we first recall that if $f = f(u)$ and $u = u(x)$, the formula of Faa di Bruno [2] for the derivatives of a composite function says that

$$\frac{d^\nu}{dx^\nu} f(u(x)) = \sum_{r=1}^{\nu} \frac{d^r}{du^r} f(u) \sum_r \frac{r!}{r_1! \cdots r_\nu!} \left(\frac{u'}{1!}\right)^{r_1} \left(\frac{u''}{2!}\right)^{r_2} \cdots \left(\frac{u^{(\nu)}}{\nu!}\right)^{r_\nu}, \tag{12}$$

where the prime denotes differentiation with respect to x. We are assuming here that the derivatives on the right of (12) exist. Here $u, \dots, u^{(\nu)}$ are evaluated at x, and Σ_r is over all partitions of r as a sum of nonnegative integers,

$$r_1 + r_2 + \cdots + r_\nu = r, \tag{13}$$

such that

$$r_1 + 2r_2 + \cdots + \nu r_\nu = \nu. \tag{14}$$

Applying (12) with $f(u) = u^\gamma$ yields

$$F^{(\nu)}(x) = \sum_{r=1}^{\nu} (\gamma)^{(r)} u^{\gamma-r}(x) \sum_r \frac{r!}{r_1! \cdots r_\nu!} \left(\frac{u'(x)}{1!}\right)^{r_1} \left(\frac{u''(x)}{2!}\right)^{r_2} \cdots \left(\frac{u^{(\nu)}(x)}{\nu!}\right)^{r_\nu},$$

where $(\gamma)^{(r)} = \gamma(\gamma-1)\cdots(\gamma-r+1)$. Since $u^{(l)}(x) = O(x^{-p-l})$, it follows that

$$(u^{\gamma-r}(x))(u'(x))^{r_1}(u''(x))^{r_2} \cdots (u^{(\nu)}(x))^{r_\nu} = O(x^{-\lambda}),$$

where

$$\lambda = p(\gamma - r) + (p+1)r_1 + (p+2)r_2 + \cdots + (p+\nu)r_\nu = p\gamma + \nu$$

because of (13) and (14). This verifies (11) with $\alpha = p\gamma$.

For our purposes it is important to note that $\mathscr{F}_\alpha$ is a vector space over the complex numbers. Moreover, if $F_i \in \mathscr{F}_{\alpha_i}$, $i = 1, 2$, then $F_1 F_2 \in \mathscr{F}_{\alpha_1+\alpha_2}$.

Lemma 2. *If $F \in \mathscr{F}_\alpha$ then*

$$\Delta^\nu F(x) = O(x^{-\alpha-\nu}), \quad \nu = 0, 1, 2, \dots$$

Proof: We show that

$$|\Delta^\nu F(x)| \le K \max_{x<\xi<x+\nu} |F^{(\nu)}(\xi)|, \tag{15}$$

where K is a constant independent of F. Since $F^{(\nu)}(x) = O(x^{-\alpha-\nu})$ this implies the conclusion.

To verify (15), we note that if $x > 1$ and $r > 0$ then Taylor's theorem implies that

$$F(x+r) = \sum_{m=0}^{\nu-1} \frac{F^{(m)}(x)}{m!} r^m + \frac{F^{(\nu)}(\xi_r)}{\nu!} r^\nu,$$

where $x < \xi_r < x + r$. Since $\Delta^\nu F(x) = \sum_{r=0}^{\nu}(-1)^{r-\nu}\binom{\nu}{r}F(x+r)$, it follows that

$$\Delta^\nu F(x) = \sum_{m=0}^{\nu-1} \frac{F^{(m)}(x)}{m!}\left(\sum_{r=0}^{\nu}(-1)^{r-\nu}\binom{\nu}{r}r^m\right) + \frac{1}{\nu!}\sum_{r=0}^{\nu}(-1)^{r-\nu}\binom{\nu}{r}r^\nu F^{(\nu)}(\xi_r).$$

Since $\sum_{r=0}^{\nu}(-1)^{r-\nu}\binom{\nu}{r}r^m = 0$ for $m = 0,\dots,\nu-1$, we can now infer (15) with $K = (\sum_{r=0}^{\nu}\binom{\nu}{r}r^\nu)/\nu!$. ■

Lemma 3. *Suppose that $F \in \mathscr{F}_\alpha$. Let ν be a fixed positive integer and let t be a real number, not an integral multiple of 2π. Then*

$$\sum_{m=n}^{\infty} F(m)e^{imt} = G(n)e^{int} + O(n^{-\alpha-\nu+1}),$$

where $G \in \mathscr{F}_\alpha$ (and G depends upon ν).

Proof: We write

$$\sum_{m=n}^{\infty} F(m)e^{imt} = e^{int}\sum_{m=0}^{\infty} F(n+m)e^{imt}. \tag{16}$$

From Lemma 2, $\Delta^\nu F(n+m) = O((n+m)^{-\alpha-\nu})$; that is, there is a constant A such that $|\Delta^\nu F(n+m)| < A(n+m)^{-\alpha-\nu}$ if $n + m > 0$. Therefore, if $n > 2$,

$$\sum_{m=0}^{\infty} |\Delta^\nu F(n+m)| < A\sum_{m=0}^{\infty}\frac{1}{(n+m)^\alpha} < A\sum_{m=0}^{\infty}\int_{n+m-1}^{n+m}\frac{dx}{(x+\alpha)^\nu}$$
$$= A\int_{n-1}^{\infty}\frac{dx}{(x+\alpha)^\nu} = O(n^{-\alpha-\nu+1}).$$

Applying Lemma 1 (specifically, (8)) with $g_m = F(n+m)$ and n fixed shows that

$$\sum_{m=0}^{\infty} F(n+m)e^{imt} = G(n) + O(n^{-\alpha-\nu+1})$$

with

$$G(x) = (1-e^{it})^{-\nu}\sum_{s=0}^{\nu-1} A_s F(x+s),$$

so $G \in \mathscr{F}_\alpha$. Now (16) implies the conclusion. ■

The following theorem shows that Theorem 4 has nontrivial applications for every positive integer q.

Theorem 5. *Suppose that*

$$a_n = f(n)e^{in\theta}, \quad n = 1,2,3,\dots, \tag{17}$$

where $f \in \mathscr{F}_\gamma$ for some $\gamma \in (0,1]$, and let q be the smallest integer such that

$$(q+1)\gamma > 1. \tag{18}$$

Then the infinite product $P = \prod^\infty(1 + a_n)$ converges if θ is not of the form $2k\pi/r$ with k an integer and $r \in \{1,\dots,q\}$.

Proof: We show by finite induction on p that if $p = 1, \dots, q$ then

$$a_n a_n^{(p)} = f_p(n) e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p}) \tag{19}$$

where $f_p \in \mathscr{F}_{(p+1)\gamma}$. In particular, (19) with $p = q$ implies that $a_n a_n^{(q)} = O(n^{-(q+1)\gamma})$, so (18) implies (5) and P converges, by Theorem 4.

From (17) and Lemma 3 with $t = \theta$, $F = f$, $\alpha = \gamma$, and $\nu = q$,

$$a_n^{(1)} = \sum_{m=n}^{\infty} f(m) e^{im\theta} = G_1(n) e^{in\theta} + O(n^{-\gamma - q + 1}),$$

with $G_1 \in \mathscr{F}_\gamma$. Therefore $a_n a_n^{(1)} = f(n) e^{in\theta}(G_1(n) e^{in\theta} + O(n^{-\gamma - q + 1}))$. Since $f \in \mathscr{F}_\gamma$, this can be rewritten as $a_n a_n^{(1)} = f_1(n) e^{2in\theta} + O(n^{-2\gamma - q + 1})$, with $f_1 = fG_1 \in \mathscr{F}_{2\gamma}$. This establishes (19) with $p = 1$, so we are finished if $q = 1$.

Now suppose that $q > 1$ and (19) holds if $1 \le p < q$. Since $(p+1)\theta$ is by assumption not an integral multiple of 2π, Lemma 3 with $t = (p+1)\theta$, $F = f_p$, $\alpha = (p+1)\gamma$, and $\nu = q - p$ implies that

$$\sum_{m=n}^{\infty} f_p(m) e^{i(p+1)m\theta} = G_p(n) e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1}),$$

where $G_p \in \mathscr{F}_{(p+1)\gamma}$. This and (19) imply that

$$a_n^{(p+1)} \equiv \sum_{m=n}^{\infty} a_m a_m^{(p)} = G_p(n) e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1}),$$

so

$$a_n a_n^{(p+1)} = f(n) e^{in\theta}\left(G_p(n) e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1})\right).$$

Since $f \in \mathscr{F}_\gamma$, this can be rewritten as

$$a_n a_n^{(p+1)} = f_{p+1}(n) e^{i(p+2)n\theta} + O(n^{-(p+2)\gamma - q + p + 1}),$$

with $f_{p+1} = fG_p \in \mathscr{F}_{(p+2)\gamma}$. This completes the induction. ■

Corollary 1. *Suppose that* $\{a_n\}^\infty$ *is as defined in Theorem 5. Then the infinite product* $\prod^\infty (1 + a_n)$ *converges if* θ *is not a rational multiple of* 2π.

Corollary 2. *Suppose that* $\alpha > 0$ *and* R *is a rational function such that* $R(x) > 0$ *on* $[N, \infty)$ $(N =$ *integer*$)$ *and* $\lim_{n \to \infty} R(x) = 0$. *Then the infinite product* $\prod_{n=N}^{\infty} (1 + (R(n))^\alpha e^{in\theta})$ *converges if* θ *is not a rational multiple of* 2π.

Corollary 3. *The infinite product* $\prod^\infty (1 + n^{-\alpha} e^{in\theta})$ *converges if* $\alpha > 0$ *and* θ *is not a rational multiple of* 2π.

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