The Gottschalk-Hedlund Theorem

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In 1955 Gottschalk and Hedlund proved that if X is a compact metric space and $T: X \to X$ is a minimal homeomorphism (in other words, if the only closed sets $Y \subset X$ for which $TY \subset Y$ are $Y = X$ and $Y = \emptyset$), and if $f: X \to \mathbf{R}$ is continuous, then $f = g - Tg$ for a continuous function g (where $Tg(x) = g(Tx)$) if and only if there exists $K < \infty$ such that $|\sum_{n=0}^{N} T^n f(x)| < K$ for all $N \in \mathbf{N}$ and $x \in X$ [**3**].

Necessity is obviously violated if one allows X to be non-compact. However, the following theorem, which for Hausdorff spaces is due to Browder [**1**], is also true:

Theorem A. *Let (X, T) be minimal, where X is any topological space and $T: X \to X$ is continuous. Let $f: X \to \mathbf{R}$ be continuous and suppose that for some $K < \infty$ we have $|\sum_{n=0}^{N} T^n f(x)| < K$ for all $N \in \mathbf{N}$ and all $x \in X$. Then for some continuous g we have $f = g - Tg$.*

In 1993 the author, at the request of P. Schwartz, produced a proof of the Gottschalk-Hedlund theorem. It amounted to a minor alteration of a proof found by the author in 1989 of the fact (due to Bohr) that the integral of an almost periodic function, if it is bounded, is itself almost periodic; see [**2**, Theorem 5.2]. The proof, which sufficed for Theorem A as well, was so innocuous as to seem hardly interesting.

Schwartz, however, noticed something novel in the proof that "made the existence of a more general cocycle theorem seem likely" [**6**]. Indeed, he was able to adapt the proof to obtain a generalization of the Gottschalk-Hedlund theorem in a setting involving convolution operators. M. Lin and V. Bergelson then suggested that the proof would go in the context of Markov operators. Schwartz obtained something along these lines in [**7**].

Finally Lin and I. Kornfeld in [**5**] obtained a more general result of this type. Let X be compact space. A Markov operator on $C(X)$ is a positive contraction T with $T1 = 1$.

Theorem B. *Let X be a compact Hausdorff space, and let T be an irreducible Markov operator on $C(X)$ (see [**5**] for the definition of irreducible). If $g \in C(X)$ satisfies $\sup_N \|\sum_{j=0}^{N} T^j g\| < \infty$, then (and only then) there exists $f \in C(X)$ with $g = f - Tf$.*

The progressively more general results obtained in [**6**], [**7**], and [**5**] suggest that perhaps the proof we present here is somewhat interesting, after all (the central idea is mimicked in all three of these cases). Some form of the proof could, of course, be distilled from any of the aforementioned papers (and, in fact, it appears explicitly in [**6**]), but it wouldn't be completely clear how to do so most simply, as there are a few complications in the more general situations that require modification of the original argument.

Since no extra effort is involved, we actually prove a version of Theorem A that allows for continuous time, in which case the functional equation takes the form $-\frac{d}{dt} g(T_t x) = f(T_t x)$; see [**4**, Lemma 2.7]. Note that the proof remains valid for maps and semiflows as well as homeomorphisms and flows. Finally, we derive as a corollary (Theorem D) the result that led to the proof's discovery: the integral of any almost periodic function over **R**, if bounded, is almost periodic.

Theorem C. *Let* $(X, \{T_t\})$ *be a minimal flow (with either discrete or continuous time). Let* $f : X \to \mathbf{R}$ *be continuous. If for all* $x \in X$ *there exists* $K < \infty$ *such that* $|\int_0^L T_t f(x)\,dt| < K$ *for all* $L \geq 0$, *then there exists a continuous function* g *such that* $-\frac{d}{dt} g(T_t x) = f(T_t x)$ *for all* $x \in X$.

Remark. In the case of discrete time, we take $\int_M^N$ to mean Σ_M^{N-1}.

Proof: For $x \in X$, put $g(x) = \sup_{N \geq 0} \int_0^N T_t f(x)\,dt$. We claim that for every $L \geq 0$, $g(x) = \sup_{N \geq L} \int_0^N T_t f(x)\,dt$.

Suppose the claim is false. Then there exists $x \in X$ and $L \geq 0$ such that $g(x) - \sup_{N \geq L} \int_0^N T_t f(x)\,dt = \epsilon > 0$. Pick $M < L$ such that $g(x) = \int_0^M T_t f(x)\,dt$. Then $\int_M^N T_t f(x)\,dt \leq -\epsilon$ for all $N \geq L$. Let $\delta = \inf_{S \geq L} \int_M^S T_t f(x)\,dt$ and fix $S \geq L$ with $\int_M^S T_t f(x)\,dt < \delta + \epsilon$.

By minimality of the flow, $\{T_t y : t \geq 0\}$ is dense in X for every $y \in X$ since its closure is a non-empty invariant set. In particular (taking $y = T_{L-M} x$), we may choose $r \geq L - M$ such that $T_r x$ lies in a neighborhood of x suitably chosen so as to ensure that

$$\int_{r+M}^{r+S} T_t f(x)\,dt = \int_M^S T_t f(T_r x)\,dt < \delta + \epsilon.$$

We have used continuity of the map $x \to \int_M^S T_t f(x)\,dt$; this is a fairly routine exercise. Hence

$$\int_M^{r+S} T_t f(x)\,dt = \int_M^{r+M} T_t f(x)\,dt + \int_{r+M}^{r+S} T_t f(x)\,dt < -\epsilon + \delta + \epsilon = \delta,$$

a contradiction that proves the claim.

We now have, for all $L \geq 0$,

$$g(T_L x) = \sup_{N \geq 0} \int_0^N T_t f(T_L x)\,dt = \sup_{N \geq 0} \int_L^{L+N} T_t f(x)\,dt = g(x) - \int_0^L T_t f(x)\,dt. \tag{1}$$

By the fundamental theorem of calculus, $-\frac{d}{dt} g(T_t x) = f(T_t x)$ (the discrete case follows directly from (1) by letting $L = 1$). All that remains, therefore, is to show that g is continuous.

By inspection g is lower semicontinuous. Let $h(x) = \inf_{N \geq 0} \int_0^N T_t f(x)\,dt$. Then

$$h(T_L x) = h(x) - \int_0^L T_t f(x)\,dt. \tag{2}$$

To see this, just replace f by $-f$ in the argument above. Clearly h is upper semi-continuous. Combining (1) and (2), we obtain $(g - h)(T_L x) = (g - h)(x)$ for all $L \geq 0$ and all $x \in X$. Let $x, y \in X$ and $\epsilon > 0$ be arbitrary. Notice that $(g - h)$ is lower semicontinuous. Utilizing the denseness of orbits, we obtain $L > 0$ such that $T_L x$ is "close enough" to y to ensure that $(g - h)(x) = (g - h)(T_L x) \geq (g - h)(y) - \epsilon$. However, since ϵ is arbitrary, $(g - h)(x) \geq (g - h)(y)$. Reversing the roles of x and y, we see that actually $(g - h)(x) = (g - h)(y)$, which implies that $(g - h)$ is constant on X. Upper semicontinuity of g now follows from upper semicontinuity of h, since g differs from h by a constant. ∎

Recall that a continuous function $f:[0,\infty) \to \mathbf{R}$ is *almost periodic* if for every $\epsilon > 0$ the set $\{n: \|f(x) - f(x+n)\|_u < \epsilon\}$ is *syndetic*, i.e. has bounded gaps. For $L \geq 0$, let $T_L f(x) = f(x + L)$. It may easily be shown that if f is almost periodic then $X = \overline{\{T_L f: L \geq 0\}}$ is compact in the uniform norm. Moreover $\{T_L\}_{L \geq 0}$ acts as a minimal isometric flow on X (with respect to the uniform norm) and $f(x)$ may be recovered by looking at the values at 0 of the members in the orbit of f: $f(x) = T_x f(0) = g(T_x f)$, where g is the (continuous) function that assigns to a member of X its value at 0.

More generally, if $(X, \{T_t\})$ is a minimal isometric flow on a compact space, $y \in X$ and $g: X \to R$ is continuous, the function $f(t) = g(T_t y)$ may be shown to be almost periodic. Hence the almost periodic functions are exactly those that arise (in this manner) from isometric flows on compact spaces. The following immediate corollary to Theorem C makes use of this fact.

Theorem D. *Let $F:[0,\infty) \to \mathbf{R}$ be almost periodic. If there exists $K < \infty$ such that $|\int_0^L F(t)\,dt| < K$ for all $L \geq 0$, then $H(s) = \int_0^s F(t)\,dt$ is almost periodic.*

Proof: We have an isometric flow on a compact space $(X, \{T_t\})$, a continuous function f on X, and a point $x \in X$ such that $F(t) = f(T_t x)$. The boundedness condition in the theorem is now exactly as in Theorem C. Hence there exists a continuous g on X such that $H(s) = \int_0^s T_t f(x)\,dt = g(x) - g(T_s x)$, which implies in particular that H is almost periodic. ∎

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