10756. *Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI.* Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

10757. *Proposed by Mark Kidwell, United States Naval Academy, Annapolis, MD.* Given integers $a_0, a_1, a_2, \ldots, a_n$ with $a_i \neq 0$ for $i \geq 1$, write $[a_0; a_1, a_2, \ldots, a_n]$ for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}.$$

Every positive rational number has a unique representation as $[a_0; a_1, a_2, \ldots, a_n]$ if we require that $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq n-1$, and $a_n > 1$ (we call this the *standard representation*), but it can have other representations $[b_0; b_1, b_2, \ldots, b_m]$ if we permit negative values for some of the b_i or if we permit $b_m = 1$. For example, $11/3 = [3; 1, 2] = [3; 1, 1, 1] = [4; -3]$. Prove or disprove: If r is a positive rational number, $r = [a_0; a_1, a_2, \ldots, a_n]$ is the standard representation, and $r = [b_0; b_1, b_2, \ldots, b_m]$ is another representation, then $a_0 + a_1 + \cdots + a_n \leq |b_0| + |b_1| + \cdots + |b_m|$, with strict inequality if any of the b_i are negative.

10758. *Proposed by Mark Sapir, Vanderbilt University, Nashville, TN.* Prove that the sum of the (decimal) digits of 9^n cannot equal 9 when $n > 2$.

10759. *Proposed by Călin Popescu, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.* In triangle ABC, let h_a denote the altitude to the side BC and let r_a denote the exradius relative to side BC, i.e., the radius of the circle tangent to the extensions of sides AB and AC and to the side BC externally. Define h_b, h_c, r_b, and r_c correspondingly. Prove that $h_a^n r_a^n + h_b^n r_b^n + h_c^n r_c^n \leq r_a^n r_b^n + r_b^n r_c^n + r_c^n r_a^n$ for any integer n, and determine conditions for equality.

SOLUTIONS

Common Eigenvector of Commuting Matrices

10633 [1997, 975]. *Proposed by Kiran S. Kedlaya, Princeton University, Princeton, NJ.* Let S be a commuting family of n-by-n matrices over an arbitrary field. Suppose the matrices in S have a common eigenvector v, so that $Mv = \lambda_M v$ for all $M \in S$. Prove that the transposes of these matrices also have a common eigenvector with these eigenvalues, that is, a vector w satisfying $M^T w = \lambda_M w$ for all $M \in S$.

Solution by Alain Tissier, Montmermeil, France. Let K be the field. Set $\phi(M) = M - \lambda_M I$ and $\phi(S) = \{\phi(M): M \in S\}$. Thus $\phi(S)$ is a commuting family of $n \times n$ matrices over K having a common nonzero vector v such that $\phi(M)v = 0$ for all $\phi(M) \in \phi(S)$. Since $\phi(M)^T = M^T - \lambda_M I$, we have to prove only that the transposes of the matrices in $\phi(S)$ have a common nonzero vector w satisfying $\phi(M)^T w = 0$ for $\phi(M) \in \phi(S)$. Thus we may suppose that $\lambda_M = 0$ for every M.

If all matrices in S are nilpotent, then the collection of transposes is also a commuting family of nilpotent matrices. In this case there is a nonzero vector w such that $M^T w = 0$ for all $M \in S$ (section 3.3 of J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, 1972). So we may assume that not all elements of S are nilpotent.

We proceed by induction on n. When $n = 1$ all the matrices are zero, so the conclusion is true. Take $n > 1$, and suppose the result is true for h-by-h matrices for each $h < n$. Let N