The Education of a Pure Mathematician

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Characters: Stu and Denton, mathematics majors; Integrity Jane, philosophy major and auditor from Hell; Professor Class, professor of mathematics.

Setting: a university classroom, during the first days of a course called Foundations of Analysis, taught by Professor Class.

WEDNESDAY, THE FIRST DAY

PROFESSOR CLASS Good morning. I hope everyone enjoyed a rewarding and relaxing summer. I'm pleased to see so many familiar names on my class list—except for Ms Integrity Jane .... Is she here? I'm sorry, how do you pronounce your last name?

INTEGRITY Just call me Integrity. You probably don't recognize me, because I'm a philosophy major. I'm just auditing.

PROFESSOR CLASS That's an unusual name.

INTEGRITY Tell me about it.

PROFESSOR CLASS Well, welcome Integrity and welcome everyone to the Foundations of Analysis, also known fondly around here as "The Education of a Pure Mathematician". We'll be covering logic, set theory, the real numbers, and the rest of the topics listed on the syllabus I'm handing out. As we work through these topics, we will come to appreciate the roles of definitions, axioms, and logical deduction, and learn how to read, understand, and write formal proofs. In a way, this course is a kind of ceremonial rite of passage, for in passing through it, we absorb how to think and act like pure mathematicians. Everyone has a copy of the textbook? Good. Then let's begin. Yes, Integrity?

INTEGRITY Before we get started, I'd like to ask a favor of you and the rest of the class. After doing some work in the philosophy of science last year, I signed up to audit this course because I thought the foundations of analysis would offer a paradigm for how scientists should build a field of rational and unbiased inquiry.

PROFESSOR CLASS I think you've come to the right place. If you can't find rational and unbiased inquiry in mathematics, where can you find it?

INTEGRITY Exactly. So I was wondering, what if we, all of us together, agreed on a short list of basic principles for the construction of a field of scientific inquiry. Then, as the course goes along, we can keep track of how consistent we're being with our basic principles. Would this be OK with everybody?

STU Sure, fine with me, why not.

DENTON Sounds like fun.

PROFESSOR CLASS I think it's a splendid idea. Does anyone object? No one? Looks like you have a deal, Integrity.

INTEGRITY After giving this some thought last night, I even have some possible principles to suggest.

PROFESSOR CLASS Excellent. Why don't you write them on the board, over to the side there, and we'll discuss them.

INTEGRITY All right. Here they are:

Some Possible Principles
for the Construction of a Field of Scientific Inquiry

M Know what something means before you ask if it's true.

A Build in no clearly unwarranted assumptions.

S Move from the simple to the less simple.

PROFESSOR CLASS Only three?

INTEGRITY I thought about others, as well as some variations, but these three struck me as more basic, less open to reasonable objections. For example, the variation of Principle **A**, "*Make* no clearly unwarranted assumptions", doesn't seem to work, since we often test a claim—that the earth is flat, that $x^2 = 2$ for some rational number, or whatever—by assuming its truth *temporarily* in order to study its consequences. But this is a far cry from assuming its truth *permanently*, which would build in an unwarranted assumption, turning the assumption into a *given* that could influence, even determine, the shape of further inquiry. Also, these principles obviously aren't supposed to be sufficient or anything. I'm only proposing them tentatively as rules that should be followed as we put together any rational and unbiased field of scientific inquiry. At the very least, you would think that a scientist trained in a field of inquiry that violates some of these principles ought to be aware of this fact and be able to defend the violations.

DENTON Aren't they just common sense, though?

STU Yeah, I was hoping we'd get some interesting arguments out of this, but these principles seem spineless. Who would violate them?

PROFESSOR CLASS In any case, I'll box them in and write "save" over here so they don't get erased. And speaking of obviously correct principles, let's begin our course with a few lectures devoted to formal logic.

INTEGRITY Before we do any mathematics?

PROFESSOR CLASS Sure. It seems only reasonable to review the general rules of correct thought before we apply them to the particular area of mathematics. Now then, for us a statement will be a sentence that can be labeled true or false. In formal logic we study the truth values of complex statements that we learn how to make in precise ways from simpler statements. For example, we ... Yes, Integrity?

INTEGRITY I'm sorry to interrupt, but I'm worried that we might already be violating Principle **A** if we continue.

PROFESSOR CLASS How's that?

INTEGRITY Well, how can we be sure that logic applies to mathematics before we do any mathematics? Wouldn't that be an unwarranted assumption? I realize it may seem odd to suggest that formal logic might not preserve truth when applied to mathematical assertions, but still.... .

DENTON Be serious. Logic isn't up for debate. It just is.

INTEGRITY I *am* serious. Logic deals with statements, that is, sentences that must be true or false, independently of whether we can know them to be true or false. But until we understand the *meaning* of mathematical assertions, their particular character and what they're about, how can we know whether it's appropriate to assume that they are always either true or false? Putting it this way, it looks as if we're going against Principle **M** too.

PROFESSOR CLASS Formal logic goes all the way back to Aristotle. For over two thousand years, we have never found logic to conflict with our experience in the world around us. Of course this is hardly surprising, since formal logic merely sets out and studies the self-evident laws of correct reasoning. It deals with formal manipulations that preserve truth, no matter *what* the meaning of the statements. So it's prior to *every* science, including mathematics.

INTEGRITY Is it prior to quantum mechanics, for example? I remember from my philosophy of science class that some sort of "quantum logic" may fit the quantum world better than classical logic.[1] Anyway, the point is, what if the meaning of a mathematical assertion *precludes* its being regarded as always true or false independently of our knowing which? Then formal logic, and perhaps some of the procedures it sanctifies (such as the Law of the Excluded Middle) would not necessarily apply. After all, the world around us is finite, while mathematics is filled with infinite processes and structures. Isn't it unjustified at this point to assume that formal logic, which seems to work beautifully in this finite world, must necessarily also work in the infinite world of mathematics?[2]

PROFESSOR CLASS We *know* it works in mathematics. It's worked perfectly for centuries.

INTEGRITY But perhaps only because classical logic was *presupposed* in that mathematics, just as you were about to presuppose it here. How could logic *not* work in a mathematics where logic—and in particular the assumption that assertions must be true or false—was built into it from the start? How can we ask whether mathematical assertions are always true or false, until we know the *meaning* of mathematical assertions? When we use a logic that takes this "bivalence" as given, before we know what mathematical assertions are about, we are in clear violation of both Principle **A** and Principle **M**.

DENTON It seems to me that bivalence is just the formal reflection of something we all believe: that mathematical assertions somehow embody "eternal truths".

INTEGRITY I believe this too, but we should not allow this sort of "religious faith" to commit us to certain types of reasoning in mathematics, ahead of understanding the meaning of mathematical assertions.[3]

DENTON Hogwash. Nothing could be more clear than that bivalence applies to mathematical assertions. Take the Riemann Hypothesis. Either all the nontrivial complex zeros of the zeta function lie on the line $\sigma = 1/2$ or there are some that don't. The Riemann Hypothesis is either true or false, whether we can prove it or not.

INTEGRITY To repeat my mantra: you cannot know this for *sure* until you first decide on the meaning you wish to assign to mathematical assertions. That's Principle **M**. I think you are being deceived by metaphors taken literally, by talk "about complex zeros of the zeta function" that you interpret as being literally about mathematical objects that exist independently of us.[4] Your "certainty" that the Riemann Hypothesis must be true or false, independently of human knowledge, therefore rests on uncertain metaphysical speculation.

STU This feels backwards. If we throw logic out, how will we know if our thinking is correct? And how can we really throw it out anyway; it's built into our language.

INTEGRITY You're not saying that purely *linguistic* structures should determine the validity of *mathematical* structures, are you?[5] Any apparently real content in such a mathematics could turn out to be an illusion created by language. And if you accept classical logic as given, so that the idea of calling the validity of that logic into question becomes unintelligible, then you could even be *trapping* mathematics in this fantasy world: you might be fixing the legitimate modes of inquiry in ways that would prevent mathematicians from ever discovering that what had been taken as given might actually be unreliable![6] Surely this would be an intolerable situation.

Look, I know it seems awfully hypothetical—I mean, really, what are the chances that after we sort out the meaning of mathematical claims, we'll find that formal logic doesn't apply—but it's at least a possibility, isn't it?

PROFESSOR CLASS Strictly speaking, I think Integrity's point—that mathematics should precede logic—is well taken, for the transformations that preserve the truth of mathematical assertions could conceivably depend on the meaning we assign to these assertions. And strictly speaking, we do not need to formalize logic as a check on our reasoning as we go on from here. In individual cases, we can still think carefully and clearly about our assumptions and procedures to check whether our argument is correct. Common sense tells us that an argument so intricate that it cannot be checked informally, cannot be checked formally either.[7] So let's skip our description of formal logic, for the moment. We can come back to it later.

Why don't we move on then to an informal description of set theory. All of mathematics rests ultimately on set theory, in the sense that every true statement in mathematics can be reduced in principle to a statement about sets that can itself be derived from the axioms of set theory.

DENTON If sets are so basic, why not give us more than an "informal" description? This is a foundations course, after all. Give us the real stuff; we can take it.

PROFESSOR CLASS I appreciate your enthusiasm, Denton, but taking up the axioms seriously would really take a big bite out of our term. As a compromise, though, let's write down some of the axioms[8]—they're called the Zermelo-Fraenkel Axioms —and we can talk about them.

AXIOM SCHEMA OF COMPREHENSION *For any property* $P(x)$ *of* x *and any* A, *there is some* B *with* $x \in B$ *if and only if* $x \in A$ *and* $P(x)$ *holds.*

AXIOM OF PAIR *Given any* A *and* B, *there is a* C *such that* $x \in C$ *if and only if* $x = A$ *or* $x = B$.

AXIOM OF INFINITY *An inductive set exists.*

AXIOM SCHEMA OF REPLACEMENT *Suppose* $P(x, y)$ *is a property such that for every* x *there is a unique* y *that makes* $P(x, y)$ *hold. Then for every* A *there is some* B *such that for every* $x \in A$ *there is some* $y \in B$ *that makes* $P(x, y)$ *hold.*

INTEGRITY Shouldn't we expect axioms to be self-evident? Or at least simpler than what we derive from them?

PROFESSOR CLASS Well, these axioms become more familiar and plausible the more you work with them. This is even true when we write the axioms more rigorously. The replacement scheme axiom, for instance, could have been written this way:

Given any formula ϕ *with free variables among* $x, y, A, w_1, \ldots, w_n$,

$$\forall A\ \forall w_1, \ldots, w_n [\forall x \in A \exists !\, y \phi \rightarrow \exists Y\ \forall x \in A\ \exists y \in Y \phi]$$

INTEGRITY Hm. Presumably you must define the positive integers in terms of these axioms?

PROFESSOR CLASS Yes, of course.

INTEGRITY But this is an obvious violation of Principle **S**! Surely we should not define something which is already clear, natural, and immediate, such as the positive integers and mathematical induction,[9] in terms of something that is far less self-evident, such as these Zermelo-Fraenkel axioms.[10] Let's put it to a class vote. How many of you find the positive integers and mathematical induction clear, natural, and obviously correct? How many feel the same way about these axioms?

PROFESSOR CLASS Obviously I don't disagree. It's plain that we have violated Principle **S**. But most mathematicians believe there are very good reasons for starting with the Zermelo-Fraenkel axioms rather than the positive integers. These axioms have given mathematics a solid foundation for many decades. Integrity, you have another comment?

INTEGRITY Yes, I've thought of a second objection. The axioms of set theory, if taken to be true, must be regarded as meaningful, for otherwise we cross Principle **M**. But to the extent that the axioms have meaning, they appear to commit us to some sort of Platonic conception of mathematical existence. And certainly the assumption that mathematical objects enjoy this kind of metaphysical existence must be seen as an unwarranted assumption, a matter of faith rather than evidence.[11] So we have a violation of Principle **A** as well.

PROFESSOR CLASS Of course most mathematicians do find some version of Platonism congenial.[12]

INTEGRITY As do I. But should we adopt a philosophy because we find it sympathetic, because, in its congenial way, it tells us what we want to hear? Or should we look for a philosophy that provides secure support for the foundations of mathematics? How can we ever feel secure if we base mathematics on the unwarranted assumption, on our private belief, that mathematical assertions refer to some objective reality? Even if we could prove the axioms of set theory were consistent —and I've heard that we can't—we wouldn't necessarily be able to construct a model.[13]

PROFESSOR CLASS I'm beginning to agree with Integrity that taking the set theory axioms seriously leads us into conflicts with not only Principle **S** but also either Principle **A** or Principle **M**. I'm also beginning to think that these three principles are not as spineless as we thought. However, I still feel that the principles reflect common sense, and that they should guide the construction of any field of rational and unbiased scientific inquiry. So let's continue to keep track of our violations, as well as what these violations tell us about our approach to mathematical inquiry. For now why don't we content ourselves with the following informal treatment of sets . . . .

FRIDAY

PROFESSOR CLASS I'm pleased to see everyone's still with us, after the starts and stops we had on Wednesday. Today should be smoother. Normally in this course I first introduce the real numbers axiomatically and only later go through the actual construction of the reals. But I doubt that Integrity Jane would be able to suspend her disbelief long enough for me to finish the axiomatic approach; so I have decided to give the construction now.

We start by defining each individual positive integer as follows:

$$1 \equiv \{\varnothing\}, \quad 2 \equiv \{\varnothing, 1\} = 1 \cup \{1\}, \quad 3 = \{\varnothing, 1, 2\} = 2 \cup \{2\}$$

and so on. (I can see you waving, Integrity, but let me continue for a minute.) To define the *set* N of positive integers, we use the Axiom of Infinity to ensure the existence of at least one set S satisfying the following two conditions,

(a) $1 \in S$
(b) For every x, $x \in S$ implies $x \cup \{x\} \in S$,

and then let N be the intersection of all sets satisfying (a) and (b). It is then simple to see that the Peano Postulates hold for N, including of course the Principle of Induction.[14]

Now Integrity has been waving her hand and shaking her head, because I guess we can all see violations of Principle **S**.

INTEGRITY Yes. I think this development seems formal and pretty, yet somehow empty, as if the desire for empirical meaning had been lost.[15] It's a terrible violation of Principle **S**, for, again, the positive integers and mathematical induction strike us as far more immediate and clear than set theory based on the Zermelo-Fraenkel axioms. Why don't we take the positive integers and their self-evident properties as given and build up mathematics from there? Can't we do that?

PROFESSOR CLASS Perhaps we could, Integrity, but what I'm describing has been found to be a precise and elegant way to define not just the positive integers, but also the real numbers. So let's push on. At this point, the rational numbers can be defined easily and their field and order properties checked. Consult your text for the details. Now to define the real numbers, we set up an equivalence relation in the collection of all Cauchy sequences of rational numbers,

$$(a_n) \cong (b_n) \text{ if } (a_n - b_n) \text{ converges to 0 in the rationals,}$$

and then we call the resulting equivalence classes real numbers.

INTEGRITY Can I ask why you put the Cauchy sequences into equivalence classes? Why not just say a real number *is* a Cauchy sequence of rationals and that two real numbers (a_n) and (b_n) are equal provided $(a_n - b_n)$ converges to 0 in the rationals?

PROFESSOR CLASS Most mathematicians find an equality based on identity fits their Platonic sympathies better than an equality based on a convention, as you propose.[16]

STU On the other hand we don't really lose anything, apart from some unnecessary abstraction, if we drop the equivalence classes, do we? After all, no one has a problem writing $1/3 = 2/6$ to mean, not the identity of the fractions, but that an equivalence relation is satisfied.

PROFESSOR CLASS Your point is well taken, Stu and Integrity. But to continue, we can now introduce operations and an ordering and verify that our set of real numbers forms an ordered field. We'll do some of this work during our next class, on Monday. At that time we will also prove that our construction has the following basic

> COMPLETENESS PROPERTY *Every bounded, nonempty set S of real numbers has a least upper bound.*

INTEGRITY And by "has" you mean...

PROFESSOR CLASS That some real number b exists that is a least upper bound for S.

INTEGRITY I guess I'm just not clear on what meaning you are giving to "b exists".

STU It's *totally* clear! It means that there *is* such a real number b.

DENTON In other words, the set of all least upper bounds is not empty.

INTEGRITY But "has a least upper bound", "a least upper bound exists", "there is a least upper bound", "the set of all least upper bounds is not empty"—these are all synonymous expressions. They don't explain the meaning at all. Do you mean that you possess a method that specifies a b that works?

PROFESSOR CLASS I guess that would depend on what you mean by "possess", "method", and "specifies".

INTEGRITY Well, suppose we consider for simplicity a less general completeness property—that every bounded sequence of rationals has a least upper bound among the reals—and ask whether we could write a program that, given any such sequence, would compute rational approximations to the least upper bound, to within any desired tolerance.

PROFESSOR CLASS Ahh...

INTEGRITY I don't believe that we *can* write such a program. I was thinking about this last night, and it seems that applied to any infinite sequence of 0s and 1s, this program either would prove that every entry vanishes or would exhibit an entry equal to 1. Most of the well-known unresolved problems of mathematics—the Riemann Hypothesis and the Twin-Prime Conjecture among them—could be solved by such a powerful program. No program of this scope exists, and surely no one believes one will ever be written.[17]

PROFESSOR CLASS This is really very interesting, Integrity. If we could have written this program, we could have said, to give it a name, that the least upper bound exists *constructively*. But it *doesn't* exist constructively. That's your point?

INTEGRITY Yes, but what I'm really worried about is what sort of meaning you can give to "b exists" when constructive existence has been ruled out. Is there anything other than some kind of metaphysical existence left?[18]

DENTON I'm confused. What's the problem? The number b still exists; the set of all least upper bounds is still nonempty. Whether b exists *constructively* or not is only an interesting side question.

INTEGRITY The question is, what do you *mean* when you claim that "b exists". We must be clear on meaning before we can decide truth. That's Principle **M**.

PROFESSOR CLASS I suppose we mean that it is false that every x in the reals R fails to be a least upper bound for S.

INTEGRITY OK, but what then is the meaning of this *new* statement. It doesn't explain the meaning of an assertion A to say that A means that B is true, and B is true means that C is true, and so on. At some point, we have to stop and give the meaning of one of these statements on its own terms. Now whatever meaning the statement "It is false that every x in R fails to be a least upper bound for S" may have, that meaning must reside in the conditions, defined by the statement itself, that allow us to say it is true. But these conditions are plainly not conditions that we, in general, can *recognize* as being true when in fact they *are* true. We just do not have the capacity to check each x in R to see whether it fails or not. The truth conditions, where any meaning must be lurking, therefore lie beyond us, untestable, beyond our experience and our consciousness. So how could we ever be said to have acquired or formed any understanding of what it takes for such a statement to be true, that is, any understanding of the meaning of the statement?

DENTON This is getting too heavy for me.

STU Can we do some mathematics now, please?

INTEGRITY Worse, there is no way for us to manifest or communicate whatever knowledge of the meaning of this statement we might claim to possess. And surely it can't be meaningful to claim that we have knowledge of something, even implicit knowledge, if we cannot, in some circumstances at least, *reveal* that knowledge.[19] Do you see what I mean?

PROFESSOR CLASS I'm beginning to, yes. And so...

INTEGRITY And so it appears that in general the statement "b exists" has no clear meaning, unless we take existence to be constructive.

DENTON Professor?

INTEGRITY I see only two ways out, and they're both bad. On the one hand, in a brazen violation of Principle **A**, you could posit the existence of a being with infinite powers, a being who can actually *perform* the infinite, even uncountable, searches required to give (nonconstructive) assertions such as "a least upper bound b exists" some meaning, some sharable, factual content.[20] Of course, you buy this meaning at a steep metaphysical price.

DENTON Professor Class?

INTEGRITY On the other hand, as a second fall-back position, you could claim that in fact the assertions of mathematics in general *have* no meaning, that in the end doing mathematics consists of manipulating meaningless strings of symbols. But then Principle **M** forces you to give up truth as well. This strikes me as falling down, rather than falling back, for this position makes our cherished mathematics, not an inquiry into "eternal truth", but a meaningless, formal game. And I'm sure you don't see yourself as having taught generations of students a meaningless game.

DENTON Professor Class, are you all right?

INTEGRITY I hate to say it, but this whole development of analysis has a formal beauty that is hollow and meaningless at the core. It just lacks—I don't know what to call it—perhaps *integrity* is the right word.[21]

DENTON Professor!

PROFESSOR CLASS Yes, Denton, I'm fine. I was just... lost in thought, I guess, and feeling a little strange.[22] Look, I know it's not the end of the hour, but why don't we quit early today, and I'll see you on Monday.

MONDAY

PROFESSOR CLASS In light of the serious questions that have come up in our class, courtesy of Integrity's initial proposal and her persistence in carrying it out, I felt driven over the weekend to think through the attitude I have always had (you could call it the classical attitude) toward mathematical existence and mathematical truth. It seems to me that the foundations of classical analysis have fallen apart under the gentle prodding of our three innocent-looking principles of inquiry. Either we must give up one or more of those common sense principles, or we must build up the foundations of analysis along different, perhaps more constructive, lines. Stu?

STU You know, for a while I really enjoyed sitting on the sidelines, watching the philosophical dispute here in class. But this is a mathematics class. Let's do some mathematics! A philosophical discussion concerning the nature of mathematical existence may be fun, and the position we take can certainly influence our *understanding* of mathematical assertions, but it can hardly have any relevance for the *doing* of mathematics. Proofs are still proofs; theorems are still theorems.[23]

PROFESSOR CLASS If that were the case, Stu, I would have found Integrity's questions less disturbing than I have. The truth is that your philosophical stance really *does* matter, and this has been known since Brouwer's dissertation in 1907. The classical and constructive positions on mathematical existence lead to two different kinds of mathematics: different procedures are seen as legitimate, different proofs are seen as convincing, and different assertions are seen as theorems. Certain classical statements are not even *intelligible* from the constructive point of view! Some of you have read Kuhn's *The Structure of Scientific Revolutions*? You probably thought that Kuhn's ideas couldn't apply to mathematics, but in fact I would say that the incommensurability in a shift from classical mathematics to constructive mathematics is as deep if not deeper than in any paradigm shift in physics or chemistry.[24] According to Kuhn, during a scientific revolution (and here I'm quoting) "the scientist's perception of his environment must be re-educated—in some familiar situations he must learn to see a new gestalt. After he has done so the world of his research will seem ... incommensurable with the one he had inhabited before."[25] Well that pretty much describes what happened to me this weekend, except that Kuhn's bloodless account doesn't tell you how completely disorienting and yet thrilling the process can be.

At the library on Saturday, I checked out the book *Foundations of Constructive Analysis* by Errett Bishop. Starting with the positive integers and their self-evident properties, he develops a natural and constructive version of mathematical analysis that appears to be consistent with our principles **A**, **S**, and **M**. It's what Brouwer should have done, if he'd been serious about selling his intuitionist program to the classical mathematicians.[26] Though it's out of print,[27] I received permission to copy the early chapters. Pass these copies around, please. This is your new text. Much of it will look familiar, but beware, it's a starkly different world: truth and constructive proof are one (so there's no such thing as an "unknowable truth"), mathematics precedes logic, and classical logic (in some cases) fails to preserve truth. In this world, a classically correct description of an integer—for example, that m is 0 if the Riemann Hypothesis is false and 1 if it's true—can become so much empty, meaningless talk, for to Bishop every integer can be converted in principle to decimal form by a finite, purely routine, process.[28] And every mathematical assertion ultimately reduces to a report, that if we make certain (perhaps hypothetical) computations within the positive integers, then we shall get certain results.[29] From the constructive standpoint, an assertion is true only when we are in a position to assert it, and false or absurd when being in a position to assert it would give rise to a contradiction. We can no longer say that every mathematical assertion is true or false, because clearly, for many assertions A, we are in no position to assert A nor in any position to assert that A can never be asserted. And we can no longer rely on proofs by contradiction, because knowing that "It is absurd that A is absurd" does not imply that we can assert A, for it does not imply that can necessarily effect the construction required to assert A.[30]

Well, everybody ready? We're starting the course over. We begin with the positive integers: 1, 2, 3, ...

NOTES

1 See [**1**, p. 320].

2 "Concerning the grounds for accepting logical laws... any 'justification' of such laws can be given only in terms of the adequacy of the language in which they are [embedded] to the specific tasks for which that language is employed.... Under the pressure of factual observation and norms of convenience familiar language habits may come to be revised; [so] the acceptance of logical principles as canonical need be neither on arbitrary grounds nor on grounds of their allegedly inherent authority, but on the ground that they effectively achieve certain postulated ends." Ernest Nagel in [**1**, p. 320]

3 "It seems to me that to clarify the sense of your [claim] you must again refer to metaphysical concepts: to some world of mathematical things existing independently of our knowledge.... But I repeat that mathematics ought not to depend on such notions as these. In fact all mathematicians... are convinced that in some sense mathematics bears upon eternal truths, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics." A. Heyting [**11**, p. 3]

4 The contemporary mathematician's use of language "seems to force us to choose between what are in fact two metaphorical descriptions of the manner in which pure mathematical knowledge is acquired: discovery or creation. And it strongly compels us to accept that the 'correct' answer is 'discovery' and not 'creation.' ...one will then be drawn almost immediately into a completely Platonistic conception.... However... so long as we do not fall for the idea that talk 'about statements' and 'about answers' must be taken literally as being about 'things' that stand in a certain relationship to us... there is no choice to be made." Gabriel Stolzenberg [**17**, p. 244]

"We can, after all, ask: What does it mean for a set to exist if it can perhaps never be defined? It seems clear that this existence can only be a manner of speaking, which can only lead to purely formal propositions—perhaps made up of very beautiful *words*—about objects *called* sets. But most mathematicians want mathematics to deal, ultimately, with performable computing operations and not to consist of formal propositions about objects called this or that." Thoralf Skolem in [**10**, p. 300]

5 "Suppose that a... mathematical construction has been carefully described by means of words, and then, the introspective character of the mathematical construction being ignored for a moment, its linguistic description is considered by itself and submitted to a linguistic application of classical logic. Is it then always possible to perform a languageless mathematical construction finding its expression in the logico-linguistic figure in question? A careful examination reveals that... with regard to the principle of the excluded third, except in special cases, the answer is in the negative." L. E. J. Brouwer in [**9**, p. 236–7]

6 [**17**, p. 225]

7 [**4**, p. 5]

8 Taken from [**12**]

9 "[Mathematical induction], inaccessible to analytic proof and to experiment, is the exact type of the *a priori* synthetic intuition. ... Why then is this view imposed upon us with such irresistible weight of evidence? It is because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power... ." H. Poincaré in [**1**, p. 388]

10 "Set-theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define and prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs." Skolem in [**10**, p. 299]

11 The axioms of set theory "if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent." K. Gödel, as quoted in [**9**, p. 99], in 1933.

12 "It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality." G. H. Hardy in [**8**, p. 1246].

13 "Suppose we have in some way proved, without thinking of any mathematical interpretation, that a logical system constructed from some linguistic axioms is non-contradictory.... If we then also find a mathematical interpretation of these axioms, does it follow...that such a mathematical system *exists*? But that has never been proved.... ." Brouwer in [**19**, p. 266–7]

14 Taken from [**2**, p. 79]

15 "[A] feeling for reality...ought to be preserved in even the most abstract studies." Bertrand Russell [**16**, p. 169]

16 [**3**, p. 13], [**4**, p. 12], [**5**, p. 15]

17 [**3**, p. 4–5], [**5**, p. 7–8]

18 "If 'to exist' does not mean 'to be constructed,' it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's [and more generally the constructive] program entails that we study mathematics as something simpler, more immediate than metaphysics, [as something where] 'to exist' must be synonymous with 'to be constructed.'" Heyting [**11**, p. 2]

19 [**7**, p. 225]

20 "Classical mathematics concerns itself with operations that can be carried out by God.... You may think that I am making a joke...by bringing God into the discussion. This is not true. I am doing my best to develop a secure philosophical foundation, based on meaning rather than formalistics, for current mathematical practice. The most solid foundation available at present seems to me to involve the consideration of a being with non-finite powers—call him God or whatever you will—in addition to the powers possessed by finite beings." Errett Bishop [**4**, p. 9]

21 "When I attempt to express in positive terms that quality in which contemporary mathematics is deficient,...I keep coming back to the term 'integrity.' Not the integrity of an isolated formalism that prides itself on the maintenance of its own standards of excellence, but an integrity that seeks common ground in the researches of pure mathematics, applied mathematics, and ...physics; that seeks to extract the maximum meaning from each new development; that is guided primarily by considerations of content rather than elegance and formal attractiveness; that sees to it that the mathematical representation of reality does not degenerate into a game.... ." Bishop [**4**, p. 4]

22 "To anyone who starts off inside the contemporary mathematician's belief system, the discovery that an entire component of the 'reality' of one's experience is produced by acts of acceptance as such in the domain of language use is not merely illuminating. In a literal sense, it is shattering: Once a mathematician has seen that his perception of the 'self-evident correctness' of the law of excluded middle is nothing more than the linguistic equivalent of an optical illusion, neither his practice of mathematics nor his understanding of it can ever be the same." Stolzenberg [**17**, p. 268]

23 "All philosophical differences...ought not to affect the detail of mathematics, but only the interpretation. Mathematics would be in a bad way if it could not proceed until [the philosophical disputes] had been settled." Russell in 1906, the year before Brouwer's dissertation provided evidence that a constructive position on mathematical existence changes the face of mathematics, as quoted in [**14**, p. 131–132]

24 Read [**15**] and [**17**]

25 [**13**, p. 112]

26 See [**15**]

27 Bishop's book has been born again in a somewhat expanded and altered form in [**5**]

28 [**4**, p. 8]

29 [**3**, p. 3] or [**5**, p. 5]

30 For much more detail on the consequences of the constructive standpoint, read [**3**], [**4**], [**5**], [**6**], or [**11**].

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