



Harmonic Products of Harmonic Functions: 10651

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where f is an arbitrary C^2 function on $[0, 1]$.

Solved also by Z. Ahmed & M.A. Prasad (India), J. Anglesio (France), G. L. Body (U. K.), P. Bracken (Canada), R. J. Chapman (U. K.), R. Cuculiere (France), J. Deutsch, K. P. Hart (The Netherlands), G. Keselman, J. H. Lindsey II, V. Lucic (Canada), W. A. Newcomb, M. Omarjee (France), K. Schilling, H.-J. Seiffert (Germany), P. Simeonov, N. C. Singer, I. Sofair, A. Stadler (Switzerland), A. Stenger, D. B. Tyler, J. H. van Lint (The Netherlands), J. Wimp, GCHQ Problems Group (U. K.), and the proposer.

Harmonic Products of Harmonic Functions

10651 [1998, 271]. *Proposed by W. K. Hayman, Imperial College, London, U. K.* If u_1 and u_2 are nonconstant real functions of two variables, and if u_1 , u_2 , and $u_1 u_2$ are all harmonic in a simply connected plane domain D , prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D , and a and b are real constants.

Solution by Tewodros Amdeberhan, DeVry Institute, North Brunswick, NJ. In \mathbb{R}^2 , we write w_x and w_y for $\partial w/\partial x$ and $\partial w/\partial y$. Let $f = u_1 + iv_1$. Since f is analytic, f^2 is analytic, and hence $2u_1 v_1 = \text{Im}(f^2)$ is harmonic. Since

$$\Delta(u_1 u_2) = \Delta u_1 + \Delta u_2 + 2\nabla u_1 \cdot \nabla u_2 \text{ and } \Delta(u_1 v_1) = \Delta u_1 + \Delta v_1 + 2\nabla u_1 \cdot \nabla v_1,$$

it follows from the hypotheses that both vectors ∇u_2 and ∇v_1 are orthogonal to ∇u_1 in R^2 . Thus

$$\nabla u_2 = a \nabla v_1, \tag{1}$$

for some real function $a = a(x, y)$. Consequently, $\Delta u_2 = a \Delta v_1 + \nabla a \cdot \nabla v_1$, and so

$$\nabla v_1 \cdot (a_x, a_y) = 0. \tag{2}$$

Rewriting (1) in terms of components yields $(u_2)_x = a(v_1)_x$ and $(u_2)_y = a(v_1)_y$. Differentiating with respect to y and x , respectively, we get

$$(u_2)_{xy} = a_y(v_1)_x + a(v_1)_{xy} \text{ and } (u_2)_{yx} = a_x(v_1)_y + a(v_1)_{yx}.$$

This shows that

$$\nabla v_1 \cdot (a_y, -a_x) = 0. \tag{3}$$

Combining (2) and (3) gives $\nabla a \equiv 0$, so a is a constant function. This in turn implies that $\nabla(u_2 - av_1) = \nabla u_2 - a \nabla v_1 \equiv 0$, proving that $u_2 - av_1$ is a constant.

Editorial comment. Irl C. Bivens notes that the “+ b ” may be eliminated in the statement of the problem if we are allowed to choose which harmonic conjugate v_1 of u_1 is to be used. He also notes that “simply connected” is not needed in the statement, since the other conditions of the problem imply the existence of a harmonic conjugate.

Solved also by K. F. Andersen (Canada), J. Anglesio (France), I. C. Bivens, R. J. Chapman (U. K.), R. Govindaraj (India), M. Gruber, R. Mortini (France), I. Netuka (Czech Republic), D. E. Pepper & J. Huntley, W. F. Trench, E. I. Verriest, and the proposer.

Large Values of Tangent

10656 [1998, 366]. *Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE, and Jeffrey Lagarias, AT&T Laboratories, Florham Park, NJ.*

(a) Show that there are infinitely many positive integers n such that $|\tan n| > n$.

(b) Show that there are infinitely many positive integers n such that $\tan n > n/4$.

Solution by Stephen M. Gagola, Jr., Kent State University, Kent, OH. We use the notation $\alpha = [a_0; a_1, a_2, \dots]$ to represent the continued fraction expansion of the irrational number

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}.$$