Multivariable Calculus and the Plus Topology

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Among the most subtle concepts in multivariable calculus are the concepts of continuity and differentiability of functions of two (or more) variables. These concepts are designed to tell us about the local behavior of a function near a point. Since "local" is defined by reference to the standard topology on $\mathbf{R}^2$, the definitions of continuity and differentiability must take into account the fact that a neighborhood of a point in this topology includes nearby points in all directions, not just the coordinate directions. As a result, these definitions involve limits in which a point (x, y) approaches a point (a, b), and such limits cannot be understood in terms of limits in which the variables x and y approach the limits a and b separately. This explains why, for example, differentiability of a function of two variables is not the same as existence of the two first partial derivatives.

But now suppose we are interested in studying the partial derivatives of a function. Since the partial derivatives are defined in terms of limits with respect to the independent variables separately, they cannot be thought of as giving us information about the local behavior of the function near a point—at least, not if "local" is defined by reference to the standard topology. But what if we use a different topology? Is there some topology on $\mathbf{R}^2$ that is appropriate for the study of partial derivatives, in the same way that the standard topology is appropriate for the study of continuity and differentiability? My purpose in this paper is to show that there is such a topology, and that the study of this topology can shed light on some of the subtleties of multivariable calculus.

The standard topology on $\mathbf{R}^2$ is defined by reference to ε-balls, where for any $\varepsilon > 0$ and any point $(a, b) \in \mathbf{R}^2$, the *ε-ball centered at* (a, b) is defined to be the set

$$B_\varepsilon(a, b) = \left\{(x, y) \in \mathbf{R}^2 \middle| \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon\right\}.$$

We define the *ε-plus centered at* (a, b) to be the set

$$+_\varepsilon(a, b) = \left\{(x, b) \in \mathbf{R}^2 \middle| |x - a| < \varepsilon\right\} \cup \left\{(a, y) \in \mathbf{R}^2 \middle| |y - b| < \varepsilon\right\}.$$

Of course, the reason for the name is that the set $+_\varepsilon(a, b)$ looks like a plus sign centered at (a, b), with "radius" ε; see Figure 1. We say that a set $U \subseteq \mathbf{R}^2$ is *plus-open* if for every $(a, b) \in U$ there is some $\varepsilon > 0$ such that $+_\varepsilon(a, b) \subseteq U$. It is easy to verify that the plus-open sets form a topology on $\mathbf{R}^2$, which we will call the *plus topology*. Clearly every open set is plus-open, but there are plus-open sets that are not open. For example, the set

$$A = \{(0,0)\} \cup \left\{(x, y) \in \mathbf{R}^2 \middle| |y| > 3|x|\right\} \cup \left\{(x, y) \in \mathbf{R}^2 \middle| |y| < |x|/3\right\}$$

is plus-open, but it is not open because it contains no ε-ball centered at $(0, 0)$; see Figure 2. Thus, the plus topology is strictly finer than the standard topology.

As evidence that the plus topology is the right topology for studying concepts involving limits with respect to the independent variables separately, we offer the following theorem. The theorem concerns separately continuous functions, where

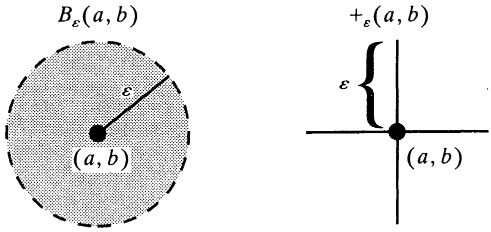

Figure 1. The sets $B_\varepsilon(a, b)$ and $+_\varepsilon(a, b)$.

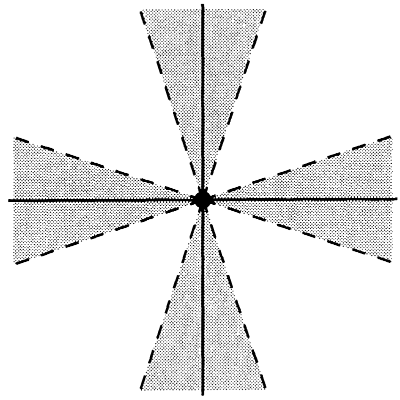

Figure 2. The set A.

a function f with domain $\mathbf{R}^2$ is called *separately continuous* if for every $b \in \mathbf{R}$, the function $f(x, b)$ is a continuous function of x, and for every $a \in \mathbf{R}$, the function $f(a, y)$ is a continuous function of y.

Theorem 1. *For every topological space Y and every function f: $\mathbf{R}^2 \to Y$, f is separately continuous if and only if it is continuous with respect to the plus topology on $\mathbf{R}^2$. Furthermore, the plus topology is the only topology for which this is true.*

Proof: Suppose that f is separately continuous, and let $V \subseteq Y$ be open. Suppose $(a, b) \in f^{-1}(V)$. Then $f(a, b) \in V$, so since the function $f(x, b)$ is continuous, there is some $\varepsilon_1 > 0$ such that if $|x - a| < \varepsilon_1$ then $f(x, b) \in V$. Similarly, there is some $\varepsilon_2 > 0$ such that if $|y - b| < \varepsilon_2$ then $f(a, y) \in V$. Clearly $+_\varepsilon(a, b) \subseteq f^{-1}(V)$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Thus $f^{-1}(V)$ is plus-open, so f is continuous with respect to the plus topology.

Now suppose that f is continuous with respect to the plus topology. Suppose that $(a, b) \in \mathbf{R}^2$, and let V be any neighborhood of $f(a, b)$ in Y. Then $(a, b) \in f^{-1}(V)$ and $f^{-1}(V)$ is plus-open, so there is some $\varepsilon > 0$ such that $+_\varepsilon(a, b) \subseteq f^{-1}(V)$. It follows that if $|x - a| < \varepsilon$ then $f(x, b) \in V$, and if $|y - b| < \varepsilon$ then $f(a, y) \in V$. Since V was arbitrary, this shows that the function $f(x, b)$ is continuous at $x = a$ and the function $f(a, y)$ is continuous at $y = b$. Thus, f is separately continuous.

Finally, to prove uniqueness, suppose that T is another topology on $\mathbf{R}^2$ with the property stated in the theorem. Let Y be $\mathbf{R}^2$ with the plus topology, and let f: $\mathbf{R}^2 \to Y$ be the identity function. The f is clearly continuous with respect to the plus topology on the domain, so by the part of the theorem already proved, f must be separately continuous. Thus, f is continuous with respect to the topology T on the domain. In other words, for every plus-open set U, $U = f^{-1}(U) \in T$, so T is at least as fine as the plus topology. Similar reasoning, with the roles of T and the plus topology reversed, shows that the plus topology is at least as fine as T, so T must be the plus topology. ∎

The plus topology is actually a special case of a kind of product topology that has appeared occasionally in the topology literature; see **[2]** and **[3]**. There are also related topologies on $\mathbf{R}^2$ that can be used to study continuity and directional derivatives in directions other than the directions of the coordinate axes. However, in this paper we restrict our attention to the plus topology on $\mathbf{R}^2$.

Corresponding to the fact that all differentiable functions are continuous, we have the following corollary of Theorem 1:

Corollary 2. *Suppose f: $\mathbf{R}^2 \to \mathbf{R}$. If the partial derivatives f_x and f_y are defined everywhere, then f is continuous with respect to the plus topology on the domain $\mathbf{R}^2$.*

Proof: If f_x and f_y are defined everywhere then f must be separately continuous, so the conclusion follows from Theorem 1. ∎

Since partial derivatives are defined using limits with respect to the independent variables separately, the first partial derivatives of a function f at a point (a, b) can be computed from the values of f at all points in any ε-plus centered at (a, b). Applying this fact at every point in a plus-open set proves our next theorem.

Theorem 3. *Suppose that f, g: $\mathbf{R}^2 \to \mathbf{R}$, U is a plus-open set, and for all $(a, b) \in U$, $f(a, b) = g(a, b)$. Then for all $(a, b) \in U$, $f_x(a, b) = g_x(a, b)$ and $f_y(a, b) = g_y(a, b)$, where each equation should be interpreted as meaning that either both partial derivatives are undefined, or both are defined and they are equal.*

Mixed higher order partial derivatives of a function f at a point (a, b) cannot be computed from the values of f on an ε-plus centered at (a, b). However, applying Theorem 3 repeatedly leads to the following corollary:

Corollary 4. *Suppose that f, g: $\mathbf{R}^2 \to \mathbf{R}$, U is a plus-open set, and for all $(a, b) \in U$, $f(a, b) = g(a, b)$. Then all partial derivatives (including all mixed partials) of f and g agree at all points in U.*

For example, consider the following two functions:

$$f(x, y) = \begin{cases} x^2 + y^2 & \text{if } (x, y) \in A \\ -1 & \text{if } (x, y) \notin A \end{cases} \qquad g(x, y) = x^2 + y^2, \tag{1}$$

where A is the plus-open set in Figure 2; see Figures 3 and 4. These functions agree at all points in A, so by Corollary 4 their partial derivatives of all orders also agree at all points in A. In particular, all partial derivatives of f and g agree at $(0, 0)$. We might say that the partial derivatives at $(0, 0)$ look at points only in a plus-open neighborhood of $(0, 0)$, and therefore they don't see the difference between f and g. But the local (in the sense of the standard topology) behavior of these functions is quite different near $(0, 0)$. For example, g is differentiable at $(0, 0)$, and f is not even continuous there. This illustrates the point that partial derivatives of a function do not give information about its local behavior.

This example also makes it clear that it is impossible to tell whether or not a function is differentiable at a particular point by examining its partial derivatives (of any order) at that point. The test for differentiability given in most multivariable calculus books says that a function is differentiable at a point if the first partial derivatives are not only defined but also *continuous* at that point. In fact, examination of the proof shows that it suffices to assume that only *one* of the partial derivatives is continuous, but this example shows why one cannot drop the continuity requirement completely.

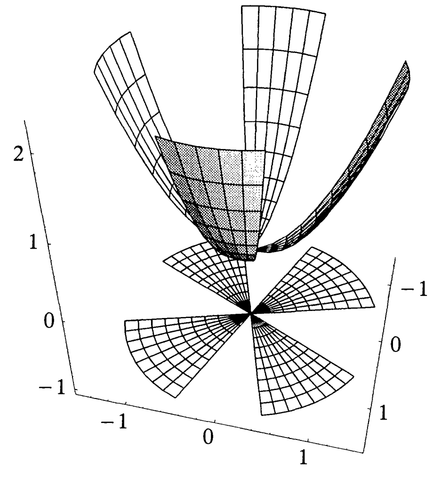

Figure 3. $z = f(x, y)$.

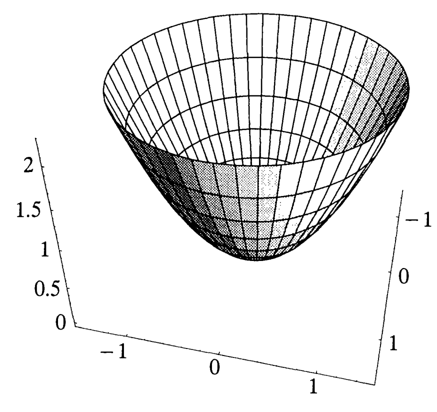

Figure 4. $z = g(x, y)$.

Here is another well-known theorem from multivariable calculus; see [**5**, p. 212]:

Theorem 5. (Second Derivative Test for Local Extrema) *Suppose that $f(x, y)$ is differentiable in a neighborhood of (a, b), $f_x(a, b) = f_y(a, b) = 0$, and f_x and f_y are differentiable at (a, b). Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. Then*:

1. *If $D > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b).*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b).*
3. *If $D < 0$ then f does not have a local extremum at (a, b).*

Once again, the plus topology can be helpful in constructing and understanding examples that illustrate why the hypotheses are needed. It is easy to check that the Second Derivative Test correctly determines that the function g in (1) has a local minimum at $(0, 0)$. Since the partial derivatives of f and g in (1) agree at $(0, 0)$, the test gives the same answer for f, even though f does not have a local minimum at $(0, 0)$. Of course, f does not satisfy the first hypothesis of Theorem 5, since it is not differentiable in a neighborhood of $(0, 0)$. But it is not hard to modify f to make it differentiable everywhere, and still have the Second Derivative Test fail. We simply need a surface that is the same as the graph of g on a plus-open neighborhood of $(0, 0)$, but is concave downward outside of that neighborhood. A natural choice would be a surface given in polar coordinates by an equation of the form $z = c(\theta)r^2$, where $c(\theta)$ is 1 when θ is close to an integer multiple of $\pi/2$ and $c(\theta)$ changes smoothly to a negative value when θ is an odd multiple of $\pi/4$. For example, we might let c be a function that is periodic with period $\pi/2$ and define $c(\theta)$ for θ between 0 and $\pi/2$ as follows:

$$c(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta \le \pi/8 \\ 1 - \exp\left[6 + \dfrac{1}{\theta - 3\pi/8} - \dfrac{1}{\theta - \pi/8}\right] & \text{if } \pi/8 < \theta < 3\pi/8 \\ 1 & \text{if } 3\pi/8 \le \theta \le \pi/2. \end{cases}$$

The graph of c is shown in Figure 5, and the surface $z = c(\theta)r^2$ is shown in Figure 6. This surface is the graph of a function $h(x, y)$ that is infinitely differentiable at all points other than the origin, since $c(\theta)$ is infinitely differentiable, and

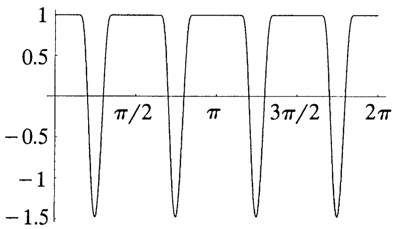

Figure 5. $y = c(\theta)$.

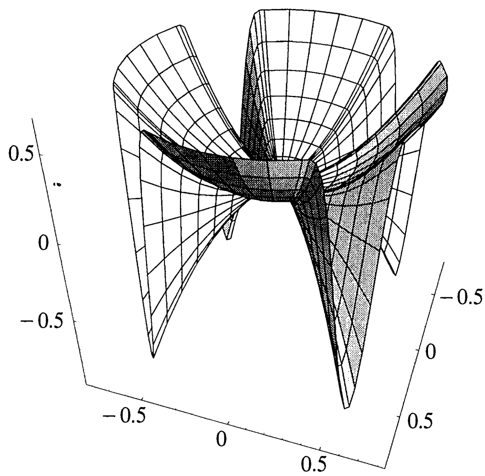

Figure 6. $z = c(\theta)r^2$.

θ and r are infinitely differentiable functions of x and y in a neighborhood of any point except the origin. Furthermore, since $c(\theta)$ is bounded, there are constants a and b such that $a(x^2 + y^2) \leq h(x, y) \leq b(x^2 + y^2)$, from which it follows that h is differentiable at $(0, 0)$. And finally, since h agrees with g on a plus-open neighborhood of $(0, 0)$, all partial derivatives of h are defined at $(0, 0)$ and are the same as the partial derivatives of g. Therefore the Second Derivative Test incorrectly indicates a local minimum for h at $(0, 0)$. The only hypothesis of Theorem 5 that we have not checked is the differentiability of the partial derivatives at $(0, 0)$, so this hypothesis must fail for h, and it cannot be dropped from the theorem. The reader might enjoy checking that $h_x(x, y) = 2xc(\theta) - yc'(\theta)$ and $h_y(x, y) = 2yc(\theta) + xc'(\theta)$. Using the fact that $c(\theta)$ and $c'(\theta)$ are bounded but not constant, it can be shown that the first partial derivatives are continuous but not differentiable at $(0, 0)$. One can get an example where the Second Derivative Test incorrectly indicates that a function does not have a local extremum by adding $z = (1 - c(\theta))r^2$ to an appropriately chosen surface with a saddle at $(0, 0)$, such as $z = x^2 + y^2 + (2 + \varepsilon)xy$, for sufficiently small positive ε. Similar examples can be found in [**4**].

All of our examples so far have been based on the plus-open set A, but there are many more exotic plus-open sets. For example, let $\{B_1, B_2, B_3, \dots\}$ be a countable basis for the standard topology on $\mathbf{R}^2$. Inductively choose, for each positive integer n, a point $(x_n, y_n) \in B_n$ such that for all $m < n$, $x_n \neq x_m$ and $y_n \neq y_m$. Let $F = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots\}$. Since F contains a point from every basic open set, $\mathbf{R}^2 \setminus F$ has empty interior in the standard topology. However, we claim that $\mathbf{R}^2 \setminus F$ is plus-open. To see why, suppose $(a, b) \in \mathbf{R}^2 \setminus F$. Then since there is at most one point in F with y-coordinate b, it is easy to find an $\varepsilon_1 > 0$ such that if $|x - a| < \varepsilon_1$ then $(x, b) \notin F$. Similarly, we can find an $\varepsilon_2 > 0$ such that if $|y - b| < \varepsilon_2$ then $(a, y) \notin F$. Thus $+_\varepsilon(a, b) \subseteq \mathbf{R}^2 \setminus F$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

Unusual plus-open sets can lead to unusual examples in multivariable calculus. For example, define j: $\mathbf{R}^2 \to \mathbf{R}$ as follows:

$$j(x, y) = \begin{cases} 1 & \text{if } (x, y) \in F \\ 0 & \text{if } (x, y) \notin F. \end{cases} \tag{2}$$

Then j agrees with the constant function $c(x, y) = 0$ on the plus-open set $\mathbf{R}^2 \setminus F$, and therefore by Corollary 4 all partial derivatives of j are defined and equal to 0 everywhere except for the countably many points in F. But since both F and $\mathbf{R}^2 \setminus F$ are dense in the plane in the standard topology, j is discontinuous everywhere.

Can the countable set of exceptional points in this example be avoided? Can a function have partial derivatives defined everywhere but be discontinuous everywhere? The answer is no, but to see why we need a fact about closures in the plus topology. For a set $X \subseteq \mathbf{R}^2$, we write $\mathrm{cl}(X)$ for the closure of X in the standard topology, and $\mathrm{cl}_+(X)$ for the closure of X in the plus topology. Note that $\mathrm{cl}_+(X) \subseteq \mathrm{cl}(X)$, since the plus topology is finer than the standard topology. For $X \subseteq \mathbf{R}$ we also write $\mathrm{cl}(X)$ for the closure of X in the standard topology on $\mathbf{R}$.

Our example $\mathbf{R}^2 \setminus F$ shows that a nonempty plus-open set can have empty interior in the standard topology. However, this cannot be true of the closure in the plus topology of a nonempty plus-open set. In fact, we have the following slightly stronger theorem, which implies that $\mathbf{R}^2$ with the plus topology is a Baire space:

Theorem 6. *Suppose U is a nonempty plus-open set, and $U = \bigcup_{n \in \mathbf{Z}^+} U_n$. Then for some n, $\mathrm{cl}_+(U_n)$ has nonempty interior in the standard topology.*

Proof: Let $(a, b) \in U$, and choose $\varepsilon > 0$ such that $+_\varepsilon(a, b) \subseteq U$. For each $x \in (a - \varepsilon, a + \varepsilon)$ and $n \in \mathbf{Z}^+$, let $Y_n^x = \{y \mid (x, y) \in U_n\}$, and let $Y^x = \bigcup_{n \in \mathbf{Z}^+} Y_n^x = \{y \mid (x, y) \in U\}$. Since $(x, b) \in +_\varepsilon(a, b) \subseteq U$ and U is plus-open, Y^x must contain an interval. Thus, by the Baire Category Theorem, there is some positive integer n_x such that $\mathrm{cl}(Y_{n_x}^x)$ contains an interval. Choose rational numbers p_x and q_x such that $p_x < q_x$ and $(p_x, q_x) \subseteq \mathrm{cl}(Y_{n_x}^x)$.

For each positive integer n and rational interval (p, q), let $X_{n,p,q} = \{x \in (a - \varepsilon, a + \varepsilon) \mid n_x = n,\ p_x = p, \text{ and } q_x = q\}$. Since there are only countably many possible values for n, p, and q, another application of the Baire Category Theorem shows that there must be some n, p, and q such that $\mathrm{cl}(X_{n,p,q})$ contains an interval. Choose $c < d$ such that $(c, d) \subseteq \mathrm{cl}(X_{n,p,q})$. For each $x \in X_{n,p,q}$, $(p, q) \subseteq \mathrm{cl}(Y_n^x)$, and it is not hard to see that therefore $X_{n,p,q} \times (p, q) \subseteq \mathrm{cl}_+(U_n)$. Similarly, since $(c, d) \subseteq \mathrm{cl}(X_{n,p,q})$, it follows that $(c, d) \times (p, q) \subseteq \mathrm{cl}_+(U_n)$, as required. ■

Using Theorem 6, we can prove the following theorem of Baire; see [**1**] and [**6**]:

Theorem 7. (Baire) *Suppose $f\colon \mathbf{R}^2 \to \mathbf{R}$ and suppose f_x and f_y are defined at all points in $\mathbf{R}^2$. Then there is a dense set of points at which f is differentiable.*

Proof: For $h \neq 0$ define functions m_h and n_h as follows:

$$m_h(x, y) = \frac{f(x + h, y) - f(x, y)}{h}, \qquad n_h(x, y) = \frac{f(x, y + h) - f(x, y)}{h}.$$

Note that m_h and n_h are separately continuous, since f is. Of course $\lim_{h \to 0} m_h(x, y) = f_x(x, y)$ and $\lim_{h \to 0} n_h(x, y) = f_y(x, y)$.

We claim first that if V is any nonempty open set and $\varepsilon > 0$ then there is a nonempty open set W such that $\mathrm{cl}(W) \subseteq V$ and for all $(u, v), (x, y) \in W$, $|f_x(u,v) - f_x(x, y)| < \varepsilon$ and $|f_y(u, v) - f_y(x, y)| < \varepsilon$. To prove the claim, first choose a nonempty open set X such that $\mathrm{cl}(X) \subseteq V$. Now for each positive integer

n and rational numbers p and q, let

$$U_{n,p,q} = \{(x,y) \in X \mid \text{for all } h, \text{ if } 0 < |h| < 1/n \text{ then } |m_h(x,y) - p| < \varepsilon/3 \text{ and } |n_h(x,y) - q| < \varepsilon/3\}.$$

Clearly $\bigcup\{U_{n,p,q} | n \in \mathbf{Z}^+ \text{ and } p, q \in \mathbf{Q}\} = X$, so by Theorem 6 we can choose $n \in \mathbf{Z}^+$ and $p, q \in \mathbf{Q}$ such that $\mathrm{cl}_+(U_{n,p,q})$ has nonempty interior in the standard topology. Let W be the interior of $\mathrm{cl}_+(U_{n,p,q})$. Then $\mathrm{cl}(W) \subseteq \mathrm{cl}(X) \subseteq V$, and using the fact that m_h and n_h are separately continuous, it is not hard to see that

$$W \subseteq \mathrm{cl}_+(U_{n,p,q}) \subseteq \{(x,y) \in \mathbf{R}^2 | \text{for all } h, \text{ if } 0 < |h| < 1/n \text{ then } |m_h(x,y) - p| \le \varepsilon/3 \text{ and } |n_h(x,y) - q| \le \varepsilon/3\}.$$

It follows that for all $(x,y) \in W$, $|f_x(x,y) - p| \le \varepsilon/3$ and $|f_y(x,y) - q| \le \varepsilon/3$, and therefore for all $(u,v),(x,y) \in W$, $|f_x(u,v) - f_x(x,y)| \le 2\varepsilon/3 < \varepsilon$ and $|f_y(u,v) - f_y(x,y)| \le 2\varepsilon/3 < \varepsilon$, as required.

Now let V_0 be any nonempty bounded open set. To prove the theorem, we must find a point in V_0 at which f is differentiable. By the claim, let V_1 be a nonempty open set such that $\mathrm{cl}(V_1) \subseteq V_0$ and for all $(u,v),(x,y) \in V_1$, $|f_x(u,v) - f_x(x,y)| < 1$ and $|f_y(u,v) - f_y(x,y)| < 1$. Applying the claim again, let V_2 be a nonempty open set such that $\mathrm{cl}(V_2) \subseteq V_1$ and for all $(u,v),(x,y) \in V_2$, $|f_x(u,v) - f_x(x,y)| < 1/2$ and $|f_y(u,v) - f_y(x,y)| < 1/2$. In general, given V_n we choose a nonempty open set V_{n+1} such that $\mathrm{cl}(V_{n+1}) \subseteq V_n$ and for all $(u,v),(x,y) \in V_{n+1}$, $|f_x(u,v) - f_x(x,y)| < 1/(n+1)$ and $|f_y(u,v) - f_y(x,y)| < 1/(n+1)$.

Let $(a,b) \in \bigcap_{n \in Z^+} V_n$. Then for every positive integer n, $(a,b) \in V_n$, and for every $(x,y) \in V_n$, $|f_x(a,b) - f_x(x,y)| < 1/n$ and $|f_y(a,b) - f_y(x,y)| < 1/n$. It follows that f_x and f_y are continuous at (a,b), and therefore f is differentiable at (a,b), as required. ■

Returning to our function j in (2), we can now see why the exceptional points cannot be avoided. If the partial derivatives of a function are defined everywhere then, by Theorem 7, it must be not only continuous but also differentiable at a dense set of points. Our function j shows that the hypotheses of Theorem 7 cannot be weakened to allow a countable set of exceptional points.

We close by mentioning two unusual properties of the plus topology that distinguish it from the standard topology on $\mathbf{R}^2$. The first follows almost immediately from Theorem 6:

Theorem 8. *The plus topology is not regular.*

Proof: We have already seen that $\mathbf{R}^2 \setminus F$ is plus-open, so F is plus-closed. Let (a,b) be any point not in F. We claim that (a,b) and F cannot be separated by plus-open sets. To see why, suppose that U and V are disjoint plus-open sets with $(a,b) \in U$ and $F \subseteq V$. Then $\mathrm{cl}_+(U)$ has empty interior in the standard topology, contradicting Theorem 6. ■

The second unusual property of the plus topology is that it is not second countable, or even first countable. In fact, it is surprisingly difficult to find a natural basis for the plus topology. Note that the sets $+_\varepsilon(a,b)$ are not plus-open, and therefore cannot be used as basis sets. It turns out that for any point $(a,b) \in \mathbf{R}^2$, any local basis at (a,b) for the plus topology must have $2^{2^{\aleph_0}}$ elements. This follows from more general results in [2].

ACKNOWLEDGMENT. I am grateful to Joan Hart, Ken Kunen, and Norton Starr for helpful comments on earlier drafts of this paper.

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