On the Generalized "Lanczos' Generalized Derivative"

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We now prove that $f \circ g$ is not Riemann integrable on [0, 1].

Let T be a division of [0, 1]. Divide T into two parts. The first part T_1 contains all the intervals in which $g(x)$ is non-zero and the second part T_2 contains the rest. The total length of all the intervals in T_1 is at most $\frac{1}{2}$; hence the total length of all the intervals in T_2 is at least $\frac{1}{2}$. But in any interval I_i of T_2, we can always find two points ξ_i and ζ_i such that $g(\xi_i) = 0$ and $g(\zeta_i) \neq 0$. Obviously, $f \circ g(\xi_i) = 0$ and $f \circ g(\zeta_i) = 1$. Thus the oscillation M_i of $f \circ g$ on I_i is 1.

Let M_α be the oscillation of $f \circ g$ on any interval I_α of T, and Δx_α be the length of the interval I_α. Then

$$\sum_{\alpha} M_\alpha \Delta x_\alpha = \sum_{T_1} M_j \Delta x_j + \sum_{T_2} M_i \Delta x_i \geq \sum_{T_2} M_i \Delta x_i = \sum_{T_2} \Delta x_i \geq \frac{1}{2}.$$

Thus $f \circ g$ is not Riemann integrable on [0, 1].

The discussion can be continued by asking for conditions on g to ensure that $f \circ g$ is Riemann integrable, provided that f is Riemann integrable. The following result provides one answer to this question. The proof is left to the reader.

Proposition 2. *Let f be a Riemann integrable function defined on $[a, b]$ and let g be a differentiable function with continuous and non-zero derivative on $[c, d]$. If the range of g is contained in $[a, b]$, then $f \circ g$ is Riemann integrable on $[c, d]$.*

REFERENCE

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This short note is an extrapolation of Groetsch's interesting article [**1**], and may lead to a clearer understanding of Lanczos' derivative. Only a minimal familiarity with random variables is required.

Lanczos' generalized derivative is defined by

$$D_h f(x) = \frac{3}{2h^3} \int_{-h}^{h} t f(x+t)\, dt,$$

where h is a parameter that can be assumed positive. It generalizes the ordinary derivative in the following two senses:

(1) Suppose $f(x)$ is locally C^4 at x_0. Then $D_h\ f(x_0) = f'(x_0) + O(h^2)$.

(2) Suppose $f(x)$ has both the right and left derivatives $f'_R(x)$ and $f'_L(x)$ at x_0. Then

$$\lim_{h\to 0} D_h f(x_0) = \frac{f'_R(x_0) + f'_L(x_0)}{2}. \tag{1}$$

A few things puzzled me as I read [**1**]. First, what does the coefficient $(3/2h^3)$ in the definition really mean? Second, how can one see easily from its integral definition that D_h is like a derivative? And finally, how exactly are the right and left derivatives involved in the limiting process of (1)? These questions gave rise to this note.

Let X be a bounded symmetric continuous random variable (i.e., X and $-X$ have the same distribution function) with variance 1. For example, X might be uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$ (with mean 0 and variance 1).

Recall that the ordinary finite difference operator d_h is defined by

$$d_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

For any positive number σ, define

$$L_\sigma f(x) = E\{X^2 d_{\sigma X} f(x)\},$$

where E is the expectation operator.

The motivation is simple. If σ is very small, $Y = \sigma X$ behaves like an atomic distribution at the origin. Therefore, one can pretend that X and Y are independent:

$$L_\sigma f(x) \simeq E\{X^2\}E\{d_Y f(x)\} = E\{d_Y f(x)\}.$$

This is an averaged d_h! Hence, L_σ does resemble the ordinary derivative for small σ.

Moreover, L_σ generalizes Lanczos' derivative D_h. To see this, take X to be any random variable that is uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$. Define $h = \sqrt{3}\,\sigma$. We show that $L_\sigma = D_h$:

$$\begin{aligned} L_\sigma f(x) &= E\left\{\frac{X}{\sigma}[f(x+\sigma X) - f(x)]\right\} = \frac{1}{\sigma}E\{Xf(x+\sigma X)\} \\ &= \frac{1}{\sigma}\int_{-\sqrt{3}}^{\sqrt{3}} tf(x+\sigma t)\frac{dt}{2\sqrt{3}} = \frac{1}{2h}\int_{-\sqrt{3}}^{\sqrt{3}} tf\left(x + \frac{h}{\sqrt{3}}t\right)dt \\ &= \frac{3}{2h^3}\int_{-h}^{h} sf(x+s)\,ds = D_h\, f(x). \end{aligned}$$

We now understand that the mysterious coefficient $3/2h^3$ has evolved from the simple parameter σ after such a long journey!

A rigorous error estimation for $L_\sigma f(x)$ follows. If $f(x)$ is C^3 near x_0, then

$$d_{\sigma X} f(x_0) = f'(x_0) + \frac{f''(x_0)}{2}\sigma X + O(\sigma^2) \quad \text{as} \quad \sigma \to 0.$$

The error term bound does not depend on the samples of X since we have assumed that X is bounded. Therefore,

$$L_\sigma f(x_0) = E\left\{X^2 f'(x_0) + \frac{f''(x_0)}{2}\sigma X^3 + X^2 O(\sigma^2)\right\} = f'(x_0) + O(\sigma^2).$$

Notice that $E\{X^3\} = 0$ since X is symmetric. This extends the first property of Lanczos' derivative.

The second property of Lanczos' derivative generalizes to L_σ in a similar fashion. Assume that both $f'_R(x_0)$ and $f'_L(x_0)$ exist. Then

$$\begin{aligned} L_\sigma f(x_0) &= E\{X^2 d_{\sigma X} f(x_0): X > 0\} + E\{X^2 d_{\sigma X} f(x_0): X < 0\} \\ &= E\{X^2 f'_R(x_0) + X^2 o(1): X > 0\} + E\{X^2 f'_L(x_0) + X^2 o(1): X < 0\} \\ &= E\{X^2 f'_R(x_0): X > 0\} + E\{X^2 f'_L(x_0): X < 0\} + o(1) \\ &= f'_R(x_0) E\{X^2: X > 0\} + f'_L(x_0) E\{X^2: X < 0\} + o(1) \\ &= \frac{f'_R(x_0) + f'_L(x_0)}{2} + o(1). \end{aligned}$$

In the last step, we have applied the symmetry condition and $E\{X^2\} = 1$. The roles of f'_R and f'_L are seen clearly from these five lines.

Finally, notice that: (1) If $f(x)$ is Lipschitz continuous at x_0 with L as its Lipschitz constant, then $|L_\sigma f(x_0)| \le L$; (2) The random variable involved can be replaced by any suitable distribution with a compact support, since we have not used the positivity condition.

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A Stability Theorem

Walter Rudin

In 1968 I proved a theorem (stated below) about zeros of holomorphic functions in a polydisc [**2**, p. 87] which was later, in [**1**], referred to, much to my surprise, as a "cornerstone" of multivariable stability theory. The authors of [**1**] pointed out, quite correctly, that my proof used quite a bit of homotopy theory, and they proceeded to prove the theorem by a sequence of more elementary steps. The present note contains an even easier proof, which is also much shorter, and which relies only on very simple properties of the index (or winding number) of a plane curve around the origin.

The following notation will be used. $\mathbf{C}$ is the complex plane, $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the set of all nonzero complex numbers, U and $\overline{U}$ are the open and closed unit discs in $\mathbf{C}$, respectively, and T is the unit circle. For $n \ge 1$,

$$\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}, \quad U^n = U \times \cdots \times U, \quad T^n = T \times \cdots \times T;$$

each of these cartesian products has n factors. The torus T^n is the so-called *distinguished boundary* of U^n; it is a small (n-dimensional) part of the whole $(2n-1)$-dimensional boundary of the polydisc U^n.

$A(U^n)$ is the class of all continuous $f: \overline{U^n} \to \mathbf{C}$ that are holomorphic in U^n.