Notice that $E\{X^3\} = 0$ since X is symmetric. This extends the first property of Lanczos' derivative.

The second property of Lanczos' derivative generalizes to L_σ in a similar fashion. Assume that both $f_R'(x_0)$ and $f_L'(x_0)$ exist. Then

$$\begin{aligned}
L_\sigma f(x_0) &= E\{X^2 d_{\sigma X} f(x_0)\colon X > 0\} + E\{X^2 d_{\sigma X} f(x_0)\colon X < 0\} \\
&= E\{X^2 f_R'(x_0) + X^2 o(1)\colon X > 0\} + E\{X^2 f_L'(x_0) + X^2 o(1)\colon X < 0\} \\
&= E\{X^2 f_R'(x_0)\colon X > 0\} + E\{X^2 f_L'(x_0)\colon X < 0\} + o(1) \\
&= f_R'(x_0) E\{X^2\colon X > 0\} + f_L'(x_0) E\{X^2\colon X < 0\} + o(1) \\
&= \frac{f_R'(x_0) + f_L'(x_0)}{2} + o(1).
\end{aligned}$$

In the last step, we have applied the symmetry condition and $E\{X^2\} = 1$. The roles of f_R' and f_L' are seen clearly from these five lines.

Finally, notice that: (1) If $f(x)$ is Lipschitz continuous at x_0 with L as its Lipschitz constant, then $|L_\sigma f(x_0)| \leq L$; (2) The random variable involved can be replaced by any suitable distribution with a compact support, since we have not used the positivity condition.

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A Stability Theorem

Walter Rudin

In 1968 I proved a theorem (stated below) about zeros of holomorphic functions in a polydisc [**2**, p. 87] which was later, in [**1**], referred to, much to my surprise, as a "cornerstone" of multivariable stability theory. The authors of [**1**] pointed out, quite correctly, that my proof used quite a bit of homotopy theory, and they proceeded to prove the theorem by a sequence of more elementary steps. The present note contains an even easier proof, which is also much shorter, and which relies only on very simple properties of the index (or winding number) of a plane curve around the origin.

The following notation will be used. $\mathbf{C}$ is the complex plane, $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the set of all nonzero complex numbers, U and $\overline{U}$ are the open and closed unit discs in $\mathbf{C}$, respectively, and T is the unit circle. For $n \geq 1$,

$$\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}, \quad U^n = U \times \cdots \times U, \quad T^n = T \times \cdots \times T;$$

each of these cartesian products has n factors. The torus T^n is the so-called *distinguished boundary* of U^n; it is a small (n-dimensional) part of the whole $(2n-1)$-dimensional boundary of the polydisc U^n.

$A(U^n)$ is the class of all continuous $f\colon \overline{U^n} \to \mathbf{C}$ that are holomorphic in U^n.

If now $\Gamma: [0, 2\pi] \to \mathbf{C}^*$ is continuous and $\Gamma(2\pi) = \Gamma(0)$ (so that $\Gamma([0, 2\pi])$ is a closed curve in $\mathbf{C}^*$) then there exists a continuous real-valued function α on $[0, 2\pi]$ such that

$$\Gamma(\theta) = |\Gamma(\theta)| \exp\{2\pi i \alpha(\theta)\} \quad (0 \leq \theta \leq 2\pi). \tag{1}$$

Since $\Gamma(2\pi) = \Gamma(0)$, $\alpha(2\pi) - \alpha(0)$ is an integer (positive, negative, or 0). This is the *index* of Γ:

$$\operatorname{Ind} \Gamma = \alpha(2\pi) - \alpha(0). \tag{2}$$

Note that $\operatorname{Ind} \Gamma$ is independent of the particular choice of α.)

We need the following properties of the index.

(I) Suppose $(s, \theta) \to \Gamma_s(\theta)$ is a continuous map from $[0,1] \times [0, 2\pi]$ into $\mathbf{C}^*$, and $\Gamma_s(2\pi) = \Gamma_s(0)$ for all s. *Then* $\operatorname{Ind} \Gamma_s$ *is the same for all* s.

The reason is simply that $\operatorname{Ind} \Gamma_s$ is a *continuous* function of s. Being integer-valued, this function is constant on the connected set $[0, 1]$.

(II) *If* $G : \overline{U} \to \mathbf{C}^*$ *is continuous and if we define* $G|_T(\theta) = G(e^{i\theta})(0 \leq \theta \leq 2\pi)$ *then* $\operatorname{Ind} G|_T = 0$.

To deduce this from (I) put $\Gamma_s(\theta) = G(se^{i\theta})$ and note that $\Gamma_1 = G|_T, \Gamma_0$ is the constant $G(0)$.

(III) *If* $h \in A(U)$ *and* $h(T) \subset \mathbf{C}^*$ *then* $\operatorname{Ind} h|_\Gamma$ *is equal to the number of zeros of* h *in* U.

This is the classical "argument principle" of complex analysis.

Theorem. *Suppose* $\Phi = (\varphi_1, \ldots, \varphi_n)$ *is a continuous map of* $\overline{U}$ *into* $\overline{U^n}$ *that carries* T *into* T^n, *such that*

$$\operatorname{Ind} \varphi_j|_T > 0 \quad \textit{for} \quad 1 \leq j \leq n \tag{3}$$

Put $K = \Phi(\overline{U})$. *Then*

$$f(T^n \cup K) = f(\overline{U^n}) \tag{4}$$

for every $f \in A(U^n)$.

Proof: Assume $f(z) \neq 0$ for every $z \in T^n \cup K$. We show that $f(z) \neq 0$ for every $z \in \overline{U^n}$. This implies the theorem, and shows why the term "stability" was used in this connection.

Fix $a = (a_1, \ldots, a_n) \in \overline{U^n}$. Let $\operatorname{Ind} \varphi_j|_T = m_j$. There exist $c_j \in \mathbf{C}$ such that

$$c_j^{m_j} = a_j \quad (1 \leq j \leq n). \tag{5}$$

Since $m_j > 0$, $|c_j| \leq 1$. Define

$$h(\lambda) = f\left(\left(\frac{\lambda + c_1}{1 + \overline{c_1}\lambda}\right)^{m_1}, \ldots, \left(\frac{\lambda + c_n}{1 + \overline{c_n}\lambda}\right)^{m_n}\right) \tag{6}$$

for $\lambda \in \overline{U}$. Then $h \in A(U)$, $h(T) \subset \mathbf{C}^*$, $h(0) = f(a)$. Hence $f(a) \neq 0$ follows from

$$\operatorname{Ind} h|_T = 0 \tag{7}$$

because of (III).

Since $(f \circ \Phi)(\overline{U}) = f(K) \subset \mathbf{C}^*$, by assumption, (II) shows that

$$\operatorname{Ind} f \circ \Phi|_T = 0. \tag{8}$$

There are continuous real-valued functions α_j, β_j such that

$$\varphi_j(e^{i\theta}) = \exp\{2\pi i\alpha_j(\theta)\}, \quad \left(\frac{e^{i\theta} + c_j}{1 + \overline{c_j}e^{i\theta}}\right)^{m_j} = \exp\{2\pi i\beta_j(\theta)\}$$

on $[0, 2\pi]$. Note that

$$\alpha_j(2\pi) - \alpha_j(0) = m_j = \beta_j(2\pi) - \beta_j(0). \tag{9}$$

Define

$$\gamma_{j,s}(\theta) = s\alpha_j(\theta) + (1-s)\beta_j(\theta) \quad (0 \le s \le 1, 0 \le \theta \le 2\pi) \tag{10}$$

and let Ψ_s: $[0, 2\pi] \to T^n$ be the map whose j^{th} component is $\exp\{2\pi i\gamma_{j,s}(\theta)\}$. Then

$$\Psi_s(2\pi) = \Psi_s(0) \quad (0 \le s \le 1), \tag{11}$$

$\Psi_s(T) \subset T^n$, hence $f(\Psi_s(T)) \subset \mathbf{C}^*$, and now (I) shows that

$$\operatorname{Ind} f \circ \Psi_1 = \operatorname{Ind} f \circ \Psi_0. \tag{12}$$

Since $f \circ \Phi|_T = f \circ \Psi_1$ and $h|_T = f \circ \Psi_0$, (12) and (8) imply (7). ■

Remarks. (i) The simplest example of a Φ as in the theorem is $\Phi(\lambda) = (\lambda, \lambda, \dots, \lambda)$. Then K is a disc (2-dimensional), $\dim(T^n \cup K) = n$, whereas $\dim \overline{U^n} = 2n$.

(ii) It is not necessary for Φ to map U into the interior U^n of $\overline{U^n}$. For example, when $n = 2$,

$$\Phi(re^{i\theta}) = \begin{cases} (2re^{i\theta}, 0) & (0 \le r \le 1/2) \\ (e^{i\theta}, 2r - 1) & (1/2 \le r \le 1) \end{cases}$$

will do nicely.

(iii) The hypothesis "$m_j > 0$ for all j" cannot be omitted. To see this, take $n = 2, \Phi(\lambda) = (\lambda, \bar{\lambda})$. Then $m_1 = 1$, $m_2 = -1$. If $f(z, w) = 1 + 4zw$, then $|f| \ge 1$ on $T^2 \cup \Phi(\overline{U})$ but $f(\frac{1}{2}, -\frac{1}{2}) = 0$.

For another example, take $\Phi(\lambda) = (\lambda, 1)$, so that $m_1 = 1$, $m_2 = 0$, and put $f(z, w) = 2w - 1$. Then $|f| \ge 1$ on $T^2 \cup \Phi(\overline{U})$ but $f(z, 1/2) = 0$ for all z.

However, the hypothesis "$m_j > 0$ for all j" *can* be replaced by "$m_j < 0$ for all j" because the theorem can then be applied to $\Phi(\bar{\lambda})$ in place of $\Phi(\lambda)$.

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